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Variational methods for the steady state response of elastic–plastic solids subjected to cyclic loads

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Abstract

Solids (or structures) of elastic–plastic internal variable material models and subjected to cyclic loads are considered. A minimum net resistant power theorem, direct consequence of the classical maximum intrinsic dissipation theorem of plasticity theory, is envisioned which describes the material behavior by determining the plastic flow mechanism (if any) corresponding to a given stress/hardening state. A maximum principle is provided which characterizes the optimal initial stress/hardening state of a cyclically loaded structure as the one such that the plastic strain and kinematic internal variable increments produced over a cycle are kinematically admissible. A steady cycle minimum principle, integrated form of the aforementioned minimum net resistant power theorem, is provided, which characterizes the structure's steady state response (steady cycle) and proves to be an extension to the present context of known principles of perfect plasticity. The optimality equations of this minimum principle are studied and two particular cases are considered: (i) loads not exceeding the shakedown limit (so recovering known results of shakedown theory) and (ii) specimen under uniform cyclic stress (or strain). Criteria to assess the structure's ratchet limit loads are given. These, together with some insensitivity features of the structure's alternating plasticity state, provide the basis to the ratchet limit load analysis problem, for which solution procedures are discussed.

Keywords: Generalized standard materials; Cyclic plasticity; Steady states; Variational methods

1. Introduction

A wide class of elastic–plastic and elastic–viscoplastic materials and structures, subjected to mechanical and/or kinematical cyclic (i.e. periodic) loads, exhibits a long-term stabilized (or steady state) response, which is independent of the initial conditions and has periodicity features like the applied load. Namely, after a transient phase lasting in general a few cycles in which no periodicity features can be recognized, the response eventually stabilizes into a *steady state* in which the stresses and the plastic strain rates turn out to be periodic. In the following, the term *steady cycle* will be used as a shorter synonym of "long-term stabilized response" and "steady state response".

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The existence of a steady cycle for materials in the mentioned class is assessed by experimental evidence (see e.g. Lemaitre and Chaboche, 1990). For cyclically loaded structures, the existence of a steady cycle was theoretically proved by Frederick and Armstrong (1966), who considered perfect plasticity and perfect viscoplasticity (see also Gokhfeld and Cherniavsky, 1980; Martin, 1975), by Mróz (1972) and Ainsworth et al. (1980), who considered kinematically hardening materials, by Halphen (1979) and Polizzotto (1994a), who considered internal variable material models. For simplicity, only time-independent plasticity will be considered in the following.

The steady cycle of a cyclically loaded structure can be directly (i.e. without the use of step-by-step analyses) determined, at least in principle, by means of a particular equation set. This equation set was explicitly reported and studied by Gokhfeld and Cherniavsky (1980), Polizzotto et al. (1990) and Polizzotto (1993a,b, 1994b) for elastic–perfectly plastic materials, by Halphen (1979) and Polizzotto (1994a) for elastic–plastic internal-variable material models, and by Polizzotto (1995) for elastic–viscoplastic materials with thermal cycling. The findings of these studies (where small displacements and strains are dealt with) are hereafter summarized for subsequent use:

(a) The stresses $\boldsymbol{\sigma}$ and the plastic strain rates $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ (with the related kinematic internal variable rates $\dot{\boldsymbol{\xi}}$) turn out to be periodic with the same period of the applied loads.
(b) The plastic strain increment over the cycle (or plastic strain ratchet), $\Delta\boldsymbol{\varepsilon}^{\mathrm{p}}$, constitutes a compatible strain field with zero displacements on the constrained part of the boundary surface of the body.
(c) The kinematic internal variables increment over the cycle, $\Delta\boldsymbol{\xi}$, turns out to vanish identically everywhere in the body.
(d) The steady state solution, that is, the solution to the mentioned set of governing equations, is unique for all, except for a time-independent residual stress field within the elastic region V_{e} (if any), where the yield limit in not attained in the steady state.

The steady cycle of a cyclically loaded structure can be categorized as follows:

(1) (*Elastic*) *shakedown*, in which the steady cycle is characterized by trivially vanishing plastic strain rates in the whole body ($V_{\mathrm{e}} = V$). This implies that plastic deformation may occur only within the transient phase, and that the structure responds to the subsequent loads in a purely elastic manner. This is the most desirable type of long-term response for structural safety, provided the amount of plastic deformation produced in the transient phase is sufficiently small.

(2) *Alternating plasticity collapse* (or *Plastic shakedown*), in which the steady cycle is characterized by zero plastic strain ratchet, $\Delta\boldsymbol{\varepsilon}^{\mathrm{p}} = \mathbf{0}$, everywhere in the body, and thus the plastic strains turn out to be periodic like the stresses. This type of long-term response, though induces low-cycle fatigue and consequently reduces the working life of the structure or specimen, is sometimes taken as a convenient basis for design purposes because of the absence of plastic strain growth.

(3) *Ratchetting* (or *Incremental collapse*), in which the steady cycle is characterized by a nonvanishing plastic strain ratchet, $\Delta\boldsymbol{\varepsilon}^{\mathrm{p}} \neq \mathbf{0}$, at least somewhere in V. This is a dangerous type of long-term response because plastic strains grow cycle by cycle, soon becoming intolerably large.

Since in general the steady state phase covers almost the entire working life of the structure, methods for the direct evaluation of the steady cycle constitute a research issue of interest in structural mechanics and engineering. After the attempts of Mróz (1972) for a variational characterization of the steady cycle, Gokhfeld and Cherniavsky (1980) formulated a maximum principle for perfectly plastic materials in the same purpose. Independently from the latter authors, Ponter and Chen (2001) formulated substantially the same principle, but shaped it in the form of minimum principle extending the classical upper bound theorem of shakedown theory to loads in excess to the shakedown limit. The essential features of the above principles consist in introducing, as additional unknowns of the classical problem, the residual stresses associated to the cycle plastic strains, and in imposing the plastic admissibility condition to the total stresses. Ponter and Chen (2001) applied their principle to evaluate the ratchet limit (see also Chen and Ponter, 2001).

The aim of the present paper is to extend the above minimum principle to internal variable material models with convex hardening potential (generalized standard materials, see Halphen and Nguyen, 1975; Lemaitre and Chaboche, 1990), exhibiting hardening saturation features. For this purpose, a particular approach is devised, which grounds on an energy principle named *minimum net resistant power theorem*, the latter being a direct consequence of the classical maximum intrinsic dissipation theorem (Halphen and Nguyen, 1975; Lemaitre and Chaboche, 1990). (For a given plastic flow mechanism, *net resistant power* equals the difference between the dissipated power and the work correspondingly done by the applied stress.) The proposed theorem proves to be a quite versatile analytical tool, since in fact it can be used not only to evaluate the plastic flow mechanism corresponding to a given stress/hardening state of the material, but also for other purposes: namely, (i) if cast in a suitable time integrated form, it can be used to determine the steady cycle in a material element, or specimen, subjected to a given cyclic stress (or strain), and (ii) if cast in a suitable time/space integrated form, it can be used to determine the steady cycle in a cyclically loaded structure. In the latter form, the proposed theorem constitutes the desired extension, to generalized standard materials, of the minimum principle given by Ponter and Chen (2001). The Euler–Lagrange equations of the related minimum problem are studied in details to show that these equations actually solve the steady cycle problem. The proposed minimum principle will be shown to incorporate ingredients both of the kinematic shakedown theorem (i.e. the so-called kinematically admissible plastic strain cycles) and of the static one (i.e. the statically admissible initial stress state); it thus can be interpreted as a special combined form of the two shakedown theorems holding for loads exceeding the shakedown limit, but which decouples into two separate shakedown statements if the applied load does not exceed the shakedown limit.

The plan of the present paper is as follows. In Section 2 the internal variable constitutive model is introduced with the related maximum intrinsic dissipation theorem. In Section 3 the minimum net resistant power theorem is established together with its ability to determine the plastic flow mechanism (if any) associated to a specified stress/hardening state of the material. In Section 4 the equation set governing the steady cycle for a cyclically loaded structure is established. Section 5 is devoted to the sequence of loading cycles, which differ from one another for the initial stress/hardening state, and provides a maximum principle for the optimal initial conditions: a classical result can in this way be found, that is, at the maximum, the plastic strain ratchet is self-compatible and the kinematic internal variable ratchet is identically vanishing. In Section 6 a minimum principle for the steady cycle in a cyclically loaded structure is presented as a consequence of the minimum net resistant power theorem, and the related Euler–Lagrange equations are studied. In Section 7 two special cases are discussed: (i) the case in which the loads do not exceed the shakedown limit (in which case the minimum principle decouples into two pieces, one of static nature is equivalent to the Melan theorem of shakedown theory, the other of kinematic nature is equivalent to the Koiter theorem of the same theory), and (ii) the case of a material element (or specimen) subjected to a given cyclic stress (or strain). In Section 8, criteria for loads at the ratchet limit are formulated for general loads and for loads not exceeding the shakedown limit. In Section 9, the ratchet limit load problem is discussed and a procedure for its evaluation together with its ground motivations are provided. Section 10 is devoted to the conclusions.

2. The material model

The material model considered here is a generalized standard material (Halphen and Nguyen, 1975; Lemaitre and Chaboche, 1990), which obeys the following equations:

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}}, \quad -\dot{\boldsymbol{\xi}} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\chi}}, \tag{1}$$

$$\dot{\lambda} \geqslant 0, \quad \dot{\lambda} f = 0, \tag{2}$$

$$f = f(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leqslant 0, \tag{3}$$

$$\boldsymbol{\sigma} = \boldsymbol{D} : \boldsymbol{\varepsilon}^{\mathrm{e}}, \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{p}} + \boldsymbol{\varepsilon}^{\theta}, \tag{4}$$

$$\boldsymbol{\chi} = \boldsymbol{\chi}(\boldsymbol{\xi}) := \frac{\partial \Psi(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}. \tag{5}$$

Here, $\boldsymbol{\sigma}$ is the stress tensor, whereas $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^{\mathrm{e}}$, $\boldsymbol{\varepsilon}^{\mathrm{p}}$ are total, elastic and plastic strain tensors, and $\boldsymbol{\varepsilon}^{\theta}$ denotes imposed thermal-like strain tensor. $\boldsymbol{D}$ is the elastic moduli fourth-order tensor of linear elasticity (with its usual symmetries and sign definiteness). $\boldsymbol{\chi}$ and $\boldsymbol{\xi}$ represent dual internal variables (respectively referred to as *static* and *kinematic*); these in practice can be scalars, or vectors, or tensors—but here are formally treated as vectors for simplicity—and are mutually related through Eq. (5), where $\Psi(\boldsymbol{\xi})$ is the hardening potential. By hypothesis, this potential is convex and such that $\|\boldsymbol{\chi}(\boldsymbol{\xi})\|$ is finitely bounded in the $\boldsymbol{\xi}$-space, that is, the material hardening states are confined within a saturation bounding surface (containing the origin $\boldsymbol{\xi} = \mathbf{0}$). Finally, $f(\boldsymbol{\sigma}, \boldsymbol{\chi})$ is the yield function, by hypothesis convex and smooth in the $(\boldsymbol{\sigma}, \boldsymbol{\chi})$-space; it also plays the role of plastic potential (associative plasticity).

As a consequence of the convexity of f, the following inequality is known to hold:

$$(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - (\boldsymbol{\chi} - \hat{\boldsymbol{\chi}}) \bullet \dot{\boldsymbol{\xi}} \geqslant 0, \tag{6}$$

where the bold dot ($\bullet$) denotes scalar product between nonCartesian vectors, the pairs $(\boldsymbol{\sigma}, \boldsymbol{\chi})$ and $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ are related with each other by Eqs. (1)–(3), whereas the pair $(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\chi}})$ is arbitrary but plastically admissible, i.e. $f(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\chi}}) \leqslant 0$. Inequality (6) is sometimes referred to as the *Druckerian inequality* because Drucker (1960) used it—though in the absence of internal variables—as an assessment of material stability. It is worth noting that the equality sign holds in (6) only if either $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \mathbf{0}$, $\dot{\boldsymbol{\xi}} = \mathbf{0}$, or $\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}$, $\boldsymbol{\chi} = \hat{\boldsymbol{\chi}}$, or both.

Inequality (6) is equivalent to the *maximum intrinsic dissipation theorem*, which more explicitly can be written:

$$\Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) = \max_{\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\chi}}}(\hat{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - \hat{\boldsymbol{\chi}} \bullet \dot{\boldsymbol{\xi}}) \quad \text{s.t. } f(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\chi}}) \leqslant 0, \tag{7}$$

where $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ is an arbitrarily fixed plastic flow mechanism and "s.t." stands for "subject to". The optimal objective function $\Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$, having the form

$$\Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) = \boldsymbol{\sigma}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - \boldsymbol{\chi}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) \bullet \dot{\boldsymbol{\xi}}, \tag{8}$$

represents the *intrinsic dissipation function*, which is convex and positively homogeneous of degree one in the $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$-space (see e.g. Martin, 1975; Kalisky, 1989; Lubliner, 1990). Φ of (8) is the amount of energy density wasted as heat in the irreversible deformation process, difference between the plastic power, $\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$, and the energy density stored in the material micro-structure, $\boldsymbol{\chi} \bullet \dot{\boldsymbol{\xi}} = \mathrm{d}\Psi/\mathrm{d}t$. The derivatives

$$\boldsymbol{\sigma} = \frac{\partial \Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})}{\partial \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}}, \quad \boldsymbol{\chi} = -\frac{\partial \Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})}{\partial \dot{\boldsymbol{\xi}}}, \tag{9}$$

(if meaningful) provide the stress $\boldsymbol{\sigma}$ and the static internal variable $\boldsymbol{\chi}$ corresponding to a given plastic mechanism $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ through the constitutive equations; more precisely, it is $f(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0$ if the given plastic mechanism is nontrivial, whereas $\boldsymbol{\sigma}$, $\boldsymbol{\chi}$ are indeterminate but plastically admissible, i.e. $f(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leqslant 0$, if $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \mathbf{0}$, $\dot{\boldsymbol{\xi}} = \mathbf{0}$, in which case the derivatives (9) lose meaning (see e.g. Martin, 1975; Lubliner, 1990). Therefore, the maximum intrinsic dissipation theorem describes the material behavior by determining the stress/hardening state corresponding, through Eqs. (1)–(3), to a given plastic flow mechanism.

3. The minimum net resistant power theorem

Assuming $\Phi(\dot{\varepsilon}, \dot{\xi})$ of Eq. (8) known, the maximum intrinsic dissipation theorem of Eq. (7) can obviously be expressed in the following form:

$$\Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) - (\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - \boldsymbol{\chi} \bullet \dot{\boldsymbol{\xi}}) \geqslant 0, \tag{10}$$

in which the pair $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ is fixed and the pair $(\boldsymbol{\sigma}, \boldsymbol{\chi})$ is any plastically admissible stress/hardening state, i.e. $f(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leqslant 0$. (The hats have been omitted for simplicity of writing.)

However, inequality (10) can evidently be read in a different alternative form stating that (10) holds true for all pairs $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ and a fixed plastically admissible pair $(\boldsymbol{\sigma}, \boldsymbol{\chi})$, with the equality sign for $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \mathbf{0}$, $\dot{\boldsymbol{\xi}} = \mathbf{0}$, or also for some nontrivial $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ when $f(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0$. With this interpretation in mind, one can realize that: if (10) is satisfied for all pairs $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ and for a fixed pair $(\boldsymbol{\sigma}, \boldsymbol{\chi})$, then necessarily $f(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leqslant 0$; conversely, if (10) is violated for some $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$, then the fixed pair $(\boldsymbol{\sigma}, \boldsymbol{\chi})$ cannot be plastically admissible, hence $f(\boldsymbol{\sigma}, \boldsymbol{\chi}) > 0$.

From this, it follows that to any given stress/hardening state, say $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}})$, one can associate the scalar quantity, here named *net resistant power*,

$$w_{\mathrm{res}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) := \Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) - (\bar{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - \bar{\boldsymbol{\chi}} \bullet \dot{\boldsymbol{\xi}}), \tag{11}$$

a function of $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ which, with its own sign behavior, provides a kinematic criterion to assess whether $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}})$ is plastically admissible or not. w_{res}, identified with the difference between the intrinsic dissipation energy and the work done by the applied stress action, can be interpreted as a measure of the work density done by the net resistant forces against the irreversible flow mechanism $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ imposed to the material element at both the macro- and micro-structure levels, namely

$$w_{\mathrm{res}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) = \left[\boldsymbol{\sigma}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) - \bar{\boldsymbol{\sigma}}\right] : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - \left[\boldsymbol{\chi}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) - \bar{\boldsymbol{\chi}}\right] \bullet \dot{\boldsymbol{\xi}}, \tag{12}$$

where $\boldsymbol{\sigma}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) = \partial\Phi/\partial\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ and $\boldsymbol{\chi}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) = -\partial\Phi/\partial\dot{\boldsymbol{\xi}}$, Eq. (9). Thus, the following statement can be phrased:

Statement 1. A *kinematic criterion* for the plastic admissibility of a given stress and hardening state of the material, say $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}})$, asserts that, if the relevant *net resistant power* $w_{\mathrm{res}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ turns out to be nonnegative for all plastic flow mechanisms $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$, then $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}})$ is plastically admissible, i.e. $f(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}}) \leqslant 0$; otherwise, if $w_{\mathrm{res}} < 0$ for some $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$, then $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}})$ is not plastically admissible, i.e. $f(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}}) > 0$. ■

The sign behavior of $w_{\mathrm{res}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$, together with its nature of being a convex one-degree positively homogeneous function in the $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$-space, suggests one to consider the following minimum problem:

$$\min_{(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})} w_{\mathrm{res}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) := \Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) - (\bar{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - \bar{\boldsymbol{\chi}} \bullet \dot{\boldsymbol{\xi}}), \tag{13}$$

by which the plastic flow mechanism $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ corresponding to a given stress/hardening state of the material can be evaluated. Essentially, two typical situations can occur in relation to the latter problem, namely:

- If $w_{\mathrm{res}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) \geqslant 0 \ \forall (\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$, a vanishing minimum exists characterized by the Kuhn–Tucker equations:

$$\bar{\boldsymbol{\sigma}} = \frac{\partial\Phi}{\partial\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}}, \quad \bar{\boldsymbol{\chi}} = -\frac{\partial\Phi}{\partial\dot{\boldsymbol{\xi}}}, \tag{14}$$

which means that $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}})$ and the optimal plastic flow mechanism $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ are mutually related by Eqs. (1)–(3)—but the derivatives in (14) become meaningless if the optimal flow mechanism is a trivial one, in which case $f(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}}) < 0$.

- If $w_{\mathrm{res}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) < 0$ for some mechanism, say $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}*}, \dot{\boldsymbol{\xi}}^*)$, no minimum can exist since in fact, in the latter case, $w_{\mathrm{res}}(c\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}*}, c\dot{\boldsymbol{\xi}}^*) = cw_{\mathrm{res}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}*}, \dot{\boldsymbol{\xi}}^*) < 0$ for any $c > 0$ and obviously $w_{\mathrm{res}}(c\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}*}, c\dot{\boldsymbol{\xi}}^*) \to -\infty$ for $c \to +\infty$. Indeed, no plastic mechanism can be associated to a pair $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}})$ violating the yield condition.

Problem (13), referred to as *minimum net resistant power theorem* in the following, describes the material behavior by determining the plastic flow mechanism (if any) which corresponds, through Eqs. (1)–(3), to a given stress/hardening state; it is the dual of the maximum intrinsic dissipation theorem. The following can thus be stated.

Statement 2. A *minimum net resistent power theorem* (dual of the maximum intrinsic dissipation theorem) holds for a material in a given stress/hardening state, say $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}})$: it states that the plastic flow mechanism $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ related to $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}})$ makes the net resistant power $w_{\mathrm{res}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ take on a vanishing minimum; conversely, if w_{res} has a vanishing minimum, the optimal pair $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ is the plastic flow mechanism related to $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}})$, and more precisely it is a nontrivial mechanism if $f(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}}) = 0$, but a trivial one if $f(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}}) < 0$. Otherwise, a *degenerate case* occurs, that is: if no plastic flow mechanism $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ can be associated to $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}})$—because the latter is not plastically admissible—then $w_{\mathrm{res}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ has no minimum; conversely, if $w_{\mathrm{res}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ has no minimum, then no plastic flow mechanism can be associated to $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}})$. ■

In the next sections, the theorem envisioned here above will be utilized—but cast in a suitably integrated form—to derive a minimum principle that characterizes the steady cycle in a solid subjected to a given cyclic load.

4. The steady cycle problem for elastic–plastic structures

A solid body, or structure, composed of elastic–plastic material like that described in Section 2, being in its initial undeformed state, occupies a region V of the three-dimensional Euclidean space and is there referred to a Cartesian orthogonal co-ordinate system $\boldsymbol{x} = (x_1, x_2, x_3)$. It is loaded by time periodic quasi-static external actions such as body forces in V, surface forces on a part, say S_{T}, of its boundary surface $S = \partial V$, imposed displacements on the complementary part of S, say $S_{\mathrm{D}} = S \setminus S_{\mathrm{T}}$, and imposed strains (e.g. thermal strains) in V. All these actions are here represented by the corresponding elastic stresses, $\boldsymbol{\sigma}^{\mathrm{E}}(\boldsymbol{x}, t)$, and displacements $\boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x}, t)$, that would arise in the structure on considering the material purely elastic. Obviously, these $\boldsymbol{\sigma}^{\mathrm{E}}$ and $\boldsymbol{u}^{\mathrm{E}}$ are periodic in time with the same period Δt of the loads. By hypothesis, the infinitesimal displacement theory is applicable and temperature variations (if any) do not affect the material data.

Let $\boldsymbol{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^{\mathrm{p}}$, $\boldsymbol{\sigma}$, ξ, $\boldsymbol{\chi}$ describe the actual elastic–plastic response of the structure to the given loads. The infinite-duration loading process consists in a sequence of equal loading cycles, all of finite duration Δt, in each of which the structure is equally loaded, but has different initial conditions dictated by the plastic strains and consequent hardening state existing in the structure at the end of the previous cycle. In the generic loading cycle, one can write:

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{\sigma}^{\mathrm{E}}(\boldsymbol{x}, t) + \boldsymbol{\rho}(\boldsymbol{x}) + \boldsymbol{\sigma}^{\mathrm{rc}}(\boldsymbol{x}, t), \tag{15}$$

$$\boldsymbol{\chi}(\boldsymbol{x}, t) = \boldsymbol{q}(\boldsymbol{x}) + \boldsymbol{\chi}^{\mathrm{c}}(\boldsymbol{x}, t), \tag{16}$$

where t denotes the cycle time, $0 \leqslant t \leqslant \Delta t$, $\boldsymbol{\sigma}^{\mathrm{rc}}$ and $\boldsymbol{\chi}^{\mathrm{c}}$ are increments of residual stresses and static internal variables produced in the course of the considered loading cycle, whereas $\boldsymbol{\rho}$ and $\boldsymbol{q}$ are the residual stresses and static internal variables at the beginning of the cycle ($t = 0$), where both $\boldsymbol{\sigma}^{\rm rc}$ and $\boldsymbol{\chi}^{\rm c}$ are identically vanishing. As far as $\Delta\boldsymbol{\sigma}^{\rm rc} := \boldsymbol{\sigma}^{\rm rc}(\boldsymbol{x}, \Delta t)$ and $\Delta\boldsymbol{\chi}^{\rm c} := \boldsymbol{\chi}^{\rm c}(\boldsymbol{x}, \Delta t)$ are nonvanishing, the subsequent loading cycle will start with $\boldsymbol{\rho}' = \boldsymbol{\rho} + \Delta\boldsymbol{\sigma}^{\rm rc}$ and $\boldsymbol{q}' = \boldsymbol{q} + \Delta\boldsymbol{\chi}^{\rm c}$ being the new updated initial conditions. The stabilized response phase will start with the first loading cycle of the cycle sequence at which $\Delta\boldsymbol{\sigma}^{\rm rc}$ and $\Delta\boldsymbol{\chi}^{\rm c}$ vanish identically to let all the subsequent cycles have equal initial conditions and thus display equal deformation processes. It follows that, in the steady state, the cycle plastic strain increment (or plastic strain ratchet), $\Delta\boldsymbol{\varepsilon}^{\rm p}$,—of which $\Delta\boldsymbol{\sigma}^{\rm rc}$ is the elastic stress response—must be a compatible strain field with zero residual displacements on $S_{\rm D}$, whereas the analogous quantity $\Delta\boldsymbol{\xi}^{\rm c}$—which induces, through the hardening law, the increment $\Delta\boldsymbol{\chi}^{\rm c} = \Delta\boldsymbol{\chi}$—must be identically vanishing.

The equations governing the deformation process in a generic loading cycle of the transient phase can be written as follows:

$$\dot{\boldsymbol{\varepsilon}}^{\rm p} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}}, \quad -\dot{\boldsymbol{\xi}} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\chi}} \quad \text{in } V \times (0, \Delta t), \tag{17}$$

$$\dot{\lambda} \geqslant 0, \quad \dot{\lambda} f = 0 \quad \text{in } V \times (0, \Delta t), \tag{18}$$

$$f = f(\boldsymbol{\sigma}^{\rm E} + \boldsymbol{\rho} + \boldsymbol{\sigma}^{\rm rc}, \boldsymbol{q} + \boldsymbol{\chi}^{\rm c}) \leqslant 0 \quad \text{in } V \times (0, \Delta t), \tag{19}$$

$$\boldsymbol{\sigma}^{\rm rc} = \boldsymbol{\sigma}^{\rm rc}(\boldsymbol{\varepsilon}^{\rm p}) := \int_V \boldsymbol{Z} : \boldsymbol{\varepsilon}^{\rm p}\, {\rm d}V' \quad \text{in } V \times (0, \Delta t), \tag{20}$$

$$\boldsymbol{\chi}^{\rm c} = \boldsymbol{\chi}^{\rm c}(\boldsymbol{\xi}^{\rm c}) := \frac{\partial \Psi^{\rm c}(\boldsymbol{\xi}^{\rm c})}{\partial \boldsymbol{\xi}^{\rm c}} \quad \text{in } V \times (0, \Delta t), \tag{21}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\chi}$ are defined by (15) and (16), and $\boldsymbol{\varepsilon}^{\rm p}$, $\boldsymbol{\xi}^{\rm c}$ are identically vanishing at $t = 0$. Moreover, $\boldsymbol{Z} = \boldsymbol{Z}(\boldsymbol{x}, \boldsymbol{x}')$ is the relevant influence tensor-valued two-point Green function giving the stress in $\boldsymbol{x}$ due to a unit strain applied at $\boldsymbol{x}'$ in the elastic body, and known to be symmetric negative semidefinite; namely, in (20), where ${\rm d}V' = {\rm d}V(\boldsymbol{x}')$, $\boldsymbol{\sigma}^{\rm rc}$ vanishes identically if $\boldsymbol{\varepsilon}^{\rm p}$ is a nontrivial self-compatible field. Also, $\Psi^{\rm c}(\boldsymbol{\xi}^{\rm c})$ of (21) is the (convex) function

$$\Psi^{\rm c}(\boldsymbol{\xi}^{\rm c}) := \Psi(\boldsymbol{p} + \boldsymbol{\xi}^{\rm c}) - \boldsymbol{q} \bullet \boldsymbol{\xi}^{\rm c}, \tag{22}$$

where $\boldsymbol{p}$ is related to $\boldsymbol{q}$ by (5), i.e. $\boldsymbol{q} = \boldsymbol{\chi}(\boldsymbol{p})$, and $\boldsymbol{\xi} = \boldsymbol{p} + \boldsymbol{\xi}^{\rm c}$, such that Eq. (21) can be recognized to be equivalent to (5); moreover, by the convexity of $\Psi^{\rm c}$, one can write:

$$\Psi^{\rm c}(\boldsymbol{\xi}^{\rm c}) \geqslant \Psi^{\rm c}(\boldsymbol{0}) = \Psi(\boldsymbol{p}) \quad \forall(\boldsymbol{p}, \boldsymbol{\xi}^{\rm c}). \tag{23}$$

It can be verified that Eqs. (15)–(21) can be uniquely solved (for instance by step-by-step integration), provided that the fields $\boldsymbol{\rho}$ and $\boldsymbol{q}$, specifying the initial stress/hardening state, are assigned everywhere in V.

The equation set (15)–(21) is valid for any loading cycle of the cycle sequence, but with $\boldsymbol{\rho}$ and $\boldsymbol{q}$ fixed at the right values for every such cycle, that is, $\boldsymbol{\rho}_{(n+1)} = \boldsymbol{\rho}_{(n)} + \Delta\boldsymbol{\sigma}^{\rm rc}_{(n)}$, $\boldsymbol{q}_{(n+1)} = \boldsymbol{q}_{(n)} + \Delta\boldsymbol{\chi}^{\rm c}_{(n)}$, where $n = 1, 2, \ldots$ denotes the loading cycle sequence and $\boldsymbol{\rho}_1, \boldsymbol{q}_1$ are known, for instance $\boldsymbol{\rho}_1 = \boldsymbol{0}$, $\boldsymbol{q}_1 = \boldsymbol{0}$ everywhere in V. A structure/loading system can be envisioned, which has the natural capacity to adjust its own initial stress/hardening state $(\boldsymbol{\rho}, \boldsymbol{q})$ such as to report itself towards a steady state, which occurs at that cycle in which $\boldsymbol{\rho}_{(n+1)} = \boldsymbol{\rho}_{(n)}$, $\boldsymbol{q}_{(n+1)} = \boldsymbol{q}_{(n)}$, hence $\Delta\boldsymbol{\sigma}^{\rm rc}_{(n)} = \boldsymbol{0}$ and $\Delta\boldsymbol{\chi}^{\rm c}_{(n)} = \boldsymbol{0}$ everywhere in V. Therefore, the problem of determining the steady cycle can be solved by adding, to Eqs. (15)–(21), besides the equilibrium conditions for $\boldsymbol{\rho}$, that is

$$\operatorname{div}\boldsymbol{\rho} = \boldsymbol{0} \quad \text{in } V, \quad \boldsymbol{\rho}\cdot\boldsymbol{n} = \boldsymbol{0} \quad \text{on } S_{\rm D}, \tag{24}$$

the equations $\Delta\boldsymbol{\sigma}^{\rm rc} = \int_V \boldsymbol{Z} : \Delta\boldsymbol{\varepsilon}^{\rm p} = \mathbf{0}$ in V and $\Delta\boldsymbol{\chi}^{\rm c} = \boldsymbol{\chi}(\boldsymbol{p} + \Delta\boldsymbol{\xi}) - \boldsymbol{q} = \mathbf{0}$ in V, which are satisfied if, and only if, $\Delta\boldsymbol{\varepsilon}^{\rm p}$ and $\Delta\boldsymbol{\xi}$ comply with the *kinematic admissibility conditions*:

$$\Delta\boldsymbol{\varepsilon}^{\rm p} := \int_0^{\Delta t} \dot{\boldsymbol{\varepsilon}}^{\rm p}\,{\rm d}t = \nabla^{\rm s}(\Delta\boldsymbol{u}^{\rm r}) \quad \text{in } V, \quad \Delta\boldsymbol{u}^{\rm r} = \mathbf{0} \quad \text{on } S_{\rm D}, \tag{25}$$

$$\Delta\boldsymbol{\xi} := \int_0^{\Delta t} \dot{\boldsymbol{\xi}}\,{\rm d}t = \mathbf{0} \quad \text{in } V, \tag{26}$$

where $\Delta\boldsymbol{u}^{\rm r}$ is some displacement field.

The latter result, derived by a simple intuitive reasoning, will be obtained by an alternative argument grounded on a maximum principle in next section.

The equation set (15)–(21) and (24)–(26), which can be used to determine the steady cycle, can be shown to admit in general a unique solution for all, except perhaps for $\boldsymbol{\rho}$, $\boldsymbol{q}$ (and $\boldsymbol{p}$) within the elastic volume $V_{\rm e}$ (if any), (Polizzotto, 1994a). This equation set takes on the following particular forms in relation to the three types of steady cycle considered in Section 1, that is:

(1) In (elastic) shakedown, the steady cycle is characterized by $\dot{\boldsymbol{\varepsilon}}^{\rm p} \equiv \mathbf{0}, \dot{\boldsymbol{\xi}} \equiv \mathbf{0}$, hence $\dot{\boldsymbol{\sigma}}^{\rm rc} \equiv \mathbf{0}, \dot{\boldsymbol{\chi}}^{\rm c} \equiv \mathbf{0}$, and thus the equation set (15)–(21), (24)–(26) loses meaning, except for the yield condition saving the form $f(\boldsymbol{\sigma}^{\rm E} + \boldsymbol{\rho}, \boldsymbol{q}) < 0$ in $V \times (0, \Delta t)$.
(2) In alternating plasticity, the steady cycle is characterized by a vanishing plastic strain ratchet, i.e. $\Delta\boldsymbol{\varepsilon}^{\rm p} = \mathbf{0}$, everywhere in V. The governing equation set remains formally unchanged, but Eq. (25) is to be replaced by the condition: $\Delta\boldsymbol{\varepsilon}^{\rm p} = \mathbf{0}$ in V.
(3) In ratchetting, the most general case, the mentioned equation set remains formally unaltered, with $\Delta\boldsymbol{\varepsilon}^{\rm p}$ being nonvanishing, at least somewhere in V.

5. Maximum principle for the cycle optimal initial conditions

Let the total intrinsic dissipation energy, $\mathscr{D}$, spent over a generic loading cycle in the transient phase, be considered, that is

$$\mathscr{D} := \int_0^{\Delta t} \int_V \Phi(\dot{\boldsymbol{\varepsilon}}^{\rm p}, \dot{\boldsymbol{\xi}})\,{\rm d}V\,{\rm d}t. \tag{27}$$

This, by (8) and (15)–(21), can be transformed as in the following:

$$\begin{aligned}\mathscr{D} = {} & \int_0^{\Delta t} \int_V \boldsymbol{\sigma}^{\rm E} : \dot{\boldsymbol{\varepsilon}}^{\rm p}\,{\rm d}V\,{\rm d}t + \int_V [\boldsymbol{\rho} : \Delta\boldsymbol{\varepsilon}^{\rm p} - \boldsymbol{q} \bullet \Delta\boldsymbol{\xi}]{\rm d}V + \frac{1}{2}\int_V \int_V \Delta\boldsymbol{\varepsilon}^{\rm p} : \boldsymbol{Z} : \Delta\boldsymbol{\varepsilon}^{\rm p}\,{\rm d}V'\,{\rm d}V \\ & - \int_V [\Psi^{\rm c}(\Delta\boldsymbol{\xi}) - \Psi^{\rm c}(\mathbf{0})]{\rm d}V,\end{aligned} \tag{28}$$

where the identity $\Delta\boldsymbol{\xi} = \Delta\boldsymbol{\xi}^{\rm c}$ has been used. The integral

$$\frac{1}{2}\int_V \int_V \Delta\boldsymbol{\varepsilon}^{\rm p} : \boldsymbol{Z} : \Delta\boldsymbol{\varepsilon}^{\rm p}\,{\rm d}V'\,{\rm d}V = -\frac{1}{2}\int_V \Delta\boldsymbol{\sigma}^{\rm rc} : \boldsymbol{D}^{-1} : \Delta\boldsymbol{\sigma}^{\rm rc}\,{\rm d}V \tag{29}$$

is obviously nonpositive. Then, using the solution to Eqs. (15)–(21) with the fields $\boldsymbol{\rho}$ and $\boldsymbol{q}$ considered arbitrarily assigned, let the following expression be builded:

$$J := \int_0^{\Delta t} \int_V \left[\Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) - \boldsymbol{\sigma}^{\mathrm{E}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \right] \mathrm{d}V \, \mathrm{d}t - \int_V [\boldsymbol{\rho} : \Delta\boldsymbol{\varepsilon}^{\mathrm{p}} - \boldsymbol{q} \bullet \Delta\boldsymbol{\xi}] \mathrm{d}V. \tag{30}$$

J turns out to be a nonpositive functional of $\boldsymbol{\rho}$ and $\boldsymbol{q}$, say $J = J[\boldsymbol{\rho}, \boldsymbol{q}]$, since in fact, by (23), (28) and (29):

$$J[\boldsymbol{\rho}, \boldsymbol{q}] = -\int_V \left[\frac{1}{2} \Delta\boldsymbol{\sigma}^{\mathrm{rc}} : \boldsymbol{D}^{-1} : \Delta\boldsymbol{\sigma}^{\mathrm{rc}} + \Psi^{\mathrm{c}}(\Delta\boldsymbol{\xi}) - \Psi^{\mathrm{c}}(\mathbf{0}) \right] \mathrm{d}V \leqslant 0, \tag{31}$$

where both $\Delta\boldsymbol{\sigma}^{\mathrm{rc}}$ and $\Delta\boldsymbol{\xi}$ depend on the fields $\boldsymbol{\rho}$ and $\boldsymbol{q}$. It follows that $J[\boldsymbol{\rho}, \boldsymbol{q}]$, being continuous and bounded from above, must admit a maximum with respect to $\boldsymbol{\rho}$ and $\boldsymbol{q}$.

Let the mechanical implications of the maximum condition for J be searched for using for this purpose the Lagrangian multiplier method. Thus, the Lagrangian functional is written by appending the constraints (24) to $-J[\boldsymbol{\rho}, \boldsymbol{q}]$, that is

$$L := -J[\boldsymbol{\rho}, \boldsymbol{q}] + \int_V \boldsymbol{v} \cdot \operatorname{div} \boldsymbol{\rho} \, \mathrm{d}V - \int_{S_{\mathrm{T}}} \boldsymbol{v} \cdot \boldsymbol{\rho} \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{32}$$

where $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x})$ is a vector-valued Lagrange multiplier. Since, at the optimum, a variation of $\boldsymbol{\rho}$ and $\boldsymbol{q}$ is expected not to produce a variation of $\dot{\boldsymbol{\epsilon}}^{\mathrm{p}}$ and $\dot{\boldsymbol{\xi}}$, the first variation of (32) with respect to $\boldsymbol{\rho}$ and $\boldsymbol{q}$ reads:

$$\delta L = \int_V \delta\boldsymbol{\rho} : [\Delta\boldsymbol{\varepsilon}^{\mathrm{p}} - \nabla^{\mathrm{s}}\boldsymbol{v}] \mathrm{d}V + \int_{S_{\mathrm{D}}} \boldsymbol{n} \cdot \delta\boldsymbol{\rho} \cdot \boldsymbol{v} \, \mathrm{d}S + \int_V \delta\boldsymbol{q} \bullet \Delta\boldsymbol{\xi} \, \mathrm{d}V. \tag{33}$$

This shows that δL is identically vanishing for arbitrary variations if, and only if, the kinematic admissibility conditions (25) and (26) are satisfied, with $\Delta\boldsymbol{u}^{\mathrm{r}}$ replaced by $\boldsymbol{v}$. At the optimum, $\Delta\boldsymbol{\sigma}^{\mathrm{rc}} = \mathbf{0}$, $\Delta\boldsymbol{\chi}^{\mathrm{c}} = \mathbf{0}$ everywhere in V by (20) and (21), hence $J_{\mathrm{opt}} = 0$ by (30).

The following statement can thus be phrased:

Statement 3. Among the set of initial self-stresses and static internal variables, $\{\boldsymbol{\rho}, \boldsymbol{q}\}$, those which generate—through Eqs. (15)–(21)—the steady cycle make J take on a vanishing maximum characterized by plastic strain and kinematical internal variable cycle increments, $\Delta\boldsymbol{\varepsilon}^{\mathrm{p}}$ and $\Delta\boldsymbol{\xi}$, being kinematically admissible, that is, satisfying (25) and (26). ■

6. The steady cycle minimum principle for a cyclically loaded structure

6.1. Formulation

The solid body, or structure, of Section 4 is considered again together with the cyclic loading there established, as well as the equation set (15)–(21) and (24)–(26) governing the structure's steady cycle. This equation set can be shown to admit a variational formulation based on a minimum principle being a direct consequence of a time/space integrated form of the minimum net resistant power theorem of Section 3. For this purpose, let the above equation set be relaxed by disregarding Eqs. (15)–(18), so obtaining a *reduced* equation set constituted by Eqs. (19)–(21) and (24)–(26).

Let one introduce the domain $\mathscr{K}$ of all sets $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}, \boldsymbol{\rho}, \boldsymbol{q})$ satisfying the above reduced equation set, that is

$$\mathscr{K} := \left\{ \begin{array}{l} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}(\boldsymbol{x}, t), \dot{\boldsymbol{\xi}}(\boldsymbol{x}, t), \boldsymbol{\rho}(\boldsymbol{x}), \boldsymbol{q}(\boldsymbol{x}), \boldsymbol{x} \in V, 0 \leqslant t \leqslant \Delta t : \\ \text{s.t. constraints} \quad (19)\text{–}(21), (24)\text{–}(26) \end{array} \right\}. \tag{34}$$

Any such set $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}, \boldsymbol{\rho}, \boldsymbol{q})$ will be referred to as *cyclically admissible solution* (CAS) in the following. A generic CAS is constituted by two parts, that is:

(i) The *kinematic part*, i.e. the fields $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}(\boldsymbol{x},t), \dot{\boldsymbol{\xi}}(\boldsymbol{x},t)$, satisfying the compatibility conditions (25) and (26)—these fields intervene in the kinematic (Koiter) theorem of shakedown theory to form a kinematically admissible plastic strain cycle.
(ii) The *static part*, i.e. the fields $\boldsymbol{\rho}(\boldsymbol{x})$, $\boldsymbol{q}(\boldsymbol{x})$, with $\boldsymbol{\rho}$ satisfying the self-equilibrium conditions (24)—these fields intervene in the static (Melan) theorem of shakedown theory to form a statically admissible initial stress/hardening state.

A generic CAS produces, through Eqs. (19)–(21), total stresses $\boldsymbol{\sigma} := \boldsymbol{\sigma}^{\mathrm{E}} + \boldsymbol{\rho} + \boldsymbol{\sigma}^{\mathrm{rc}}$ and total static internal variables, $\boldsymbol{\chi} = \boldsymbol{q} + \boldsymbol{\chi}^{\mathrm{c}}$, which are *periodic* (both $\boldsymbol{\sigma}^{\mathrm{rc}}$ and $\boldsymbol{\chi}^{\mathrm{c}}$ vanish at $t = 0$ and $t = \Delta t$) and *plastically admissible*. The latter constraint amounts to a coupled restriction upon the kinematic and static parts mentioned above.

A CAS having the kinematic part trivially vanishing, i.e. $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \mathbf{0}, \dot{\boldsymbol{\xi}} = \mathbf{0}$ in $V \times (0, \Delta t)$, hence producing total stresses and total static internal variables of the (uncoupled) form $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{E}} + \boldsymbol{\rho}$ and $\boldsymbol{\chi} = \boldsymbol{q}$, respectively, will be referred to as *trivial* CAS. Such a trivial CAS does exist in $\mathscr{K}$ if, and only if, there exists a static part $(\boldsymbol{\rho}, \boldsymbol{q})$ such that $f(\boldsymbol{\sigma}^{\mathrm{E}} + \boldsymbol{\rho}, \boldsymbol{q}) \leqslant 0$ in $V \times (0, \Delta t)$, what certainly occurs for all loads not exceeding the shakedown limit, due to the Melan theorem of shakedown theory. Since in general, in the latter case, the static part $(\boldsymbol{\rho}, \boldsymbol{q})$ is not unique, it follows that, for loads not exceeding the shakedown limit, in general $\mathscr{K}$ contains a continuous set of trivial CASs—but only one at the shakedown limit in conditions of ratchetting collapse mode (Polizzotto, 1994a).

Then, using a generic CAS—with the related stress and static internal variable increments (20) and (21)—let the following integral expression be builded, i.e.

$$W_{\mathrm{res}} := \int_0^{\Delta t} \int_V \left[\Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) - (\boldsymbol{\sigma}^{\mathrm{E}} + \boldsymbol{\rho} + \boldsymbol{\sigma}^{\mathrm{rc}}) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + (\boldsymbol{q} + \boldsymbol{\chi}^{\mathrm{c}}) \bullet \dot{\boldsymbol{\xi}} \right] \mathrm{d}V \, \mathrm{d}t \geqslant 0 \quad \text{in } \mathscr{K}, \tag{35}$$

which is a time/space integrated form of the net resistant power, $w_{\mathrm{res}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$, of Section 3, adjusted to the present context. W_{res} turns out to be nonnegative because the square-bracketed integrand itself is so as a consequence of (6) and (19), or equivalently, of (19) and the minimum net resistant power theorem—the degenerate case being excluded by (19)—. From the identities:

$$\int_V \boldsymbol{\rho} : \Delta\boldsymbol{\varepsilon}^{\mathrm{p}} \, \mathrm{d}V = 0 \quad \text{by Eqs. (24) and (25)} \tag{36}$$

$$\int_V \boldsymbol{q} \bullet \Delta\boldsymbol{\xi} \, \mathrm{d}V = 0 \quad \text{by Eq. (26),} \tag{37}$$

$$\int_0^{\Delta t} \int_V \boldsymbol{\sigma}^{\mathrm{rc}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \mathrm{d}V \mathrm{d}t = \int_V \int_V \frac{1}{2} \Delta\boldsymbol{\varepsilon}^{\mathrm{p}} : \boldsymbol{Z} : \Delta\boldsymbol{\varepsilon}^{\mathrm{p}} \, \mathrm{d}V' \, \mathrm{d}V = 0 \quad \text{by Eqs. (20) and (25),} \tag{38}$$

$$\int_0^{\Delta t} \int_V \boldsymbol{\chi}^{\mathrm{c}} \bullet \dot{\boldsymbol{\xi}} \mathrm{d}V \mathrm{d}t = \int_V [\Psi^{\mathrm{c}}(\Delta\boldsymbol{\xi}) - \Psi^{\mathrm{c}}(\mathbf{0})] \mathrm{d}V = 0 \quad \text{by Eqs. (21) and (26),} \tag{39}$$

it follows that W_{res} of (35) simplifies into

$$W_{\mathrm{res}} = \int_0^{\Delta t} \int_V \left[\Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) - \boldsymbol{\sigma}^{\mathrm{E}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \right] \mathrm{d}V \, \mathrm{d}t \geqslant 0 \quad \text{in } \mathscr{K}. \tag{40}$$

This makes it clear that W_{res} is a nonnegative functional defined over $\mathscr{K}$ and has the meaning of *total net resistant work* which, for every CAS, equals the difference between the related total intrinsic dissipation energy and the work correspondingly done by the loads. Considering that there exists in $\mathscr{K}$ a particular CAS which together with some $\dot{\lambda}(\boldsymbol{x},t)$ form the solution to the complete equation set (15)–(21), (24)–(26), and that $W_{\text{res}} = 0$ for this particular CAS, it can be stated that the functional (40) admits a vanishing minimum in $\mathscr{K}$ and that the minimum solution characterizes the structure's steady cycle.

The following minimization problem is thus pertinent being considered at this point:

$$\min_{\mathscr{K}} W_{\text{res}} := \int_0^{\Delta t} \int_V \left[\Phi(\dot{\boldsymbol{\varepsilon}}^{\text{p}}, \dot{\boldsymbol{\xi}}) - \boldsymbol{\sigma}^{\text{E}} : \dot{\boldsymbol{\varepsilon}}^{\text{p}}\right] \mathrm{d}V \, \mathrm{d}t. \tag{41}$$

Since at the minimum $W_{\text{res}} = 0$ and thus the nonnegative square-bracketed integrand of (35) must also vanish, the following identity is satisfied at the optimum:

$$\Phi(\dot{\boldsymbol{\varepsilon}}^{\text{p}}, \dot{\boldsymbol{\xi}}) = (\boldsymbol{\sigma}^{\text{E}} + \boldsymbol{\rho} + \boldsymbol{\sigma}^{\text{rc}}) : \dot{\boldsymbol{\varepsilon}}^{\text{p}} - (\boldsymbol{q} + \boldsymbol{\chi}^{\text{c}}) \bullet \dot{\boldsymbol{\xi}} \quad \text{in } V \times (0, \Delta t), \tag{42}$$

which implies

$$\boldsymbol{\sigma} := \boldsymbol{\sigma}^{\text{E}} + \boldsymbol{\rho} + \boldsymbol{\sigma}^{\text{rc}} = \frac{\partial \Phi}{\partial \dot{\boldsymbol{\varepsilon}}^{\text{p}}} \quad \text{in } V \times (0, \Delta t), \tag{43}$$

$$\boldsymbol{\chi} := \boldsymbol{q} + \boldsymbol{\chi}^{\text{c}} = -\frac{\partial \Phi}{\partial \dot{\boldsymbol{\xi}}} \quad \text{in } V \times (0, \Delta t). \tag{44}$$

That is, the optimal pairs $(\boldsymbol{\sigma}, \boldsymbol{\chi})$ and $(\dot{\boldsymbol{\varepsilon}}^{\text{p}}, \dot{\boldsymbol{\xi}})$ turn out to be mutually related by the plasticity constitutive laws (17)–(19). Therefore, the optimal CAS with some $\dot{\lambda}(\boldsymbol{x},t)$ solve the complete equation set (15)–(21) and (24)–(26). It seems appropriate to call the above principle "steady cycle minimum principle".

6.2. Study of the optimality equations

This study is required in order to better understand all the mechanical implications of the minimum problem (41). The Euler–Lagrange equations can be derived by the Lagrange multiplier method. For this purpose, appending the constraints of (41) to the related objective functional, the Lagrangian functional is obtained, that is:

$$\begin{aligned} L_{\mathscr{K}} = {} & \int_0^{\Delta t} \int_V \left[\Phi(\dot{\boldsymbol{\varepsilon}}^{\text{p}}, \dot{\boldsymbol{\xi}}) - \boldsymbol{\sigma}^{\text{E}} : \dot{\boldsymbol{\varepsilon}}^{\text{p}}\right] \mathrm{d}V \, \mathrm{d}t + \int_0^{\Delta t} \int_V \dot{l} f(\boldsymbol{\sigma}, \boldsymbol{\chi}) \, \mathrm{d}V \, \mathrm{d}t \\ & + \int_0^{\Delta t} \int_V \dot{\boldsymbol{e}}^{\text{p}} : \left[-\boldsymbol{\sigma}^{\text{rc}} + \int_V \boldsymbol{Z} : \boldsymbol{\varepsilon}^{\text{p}} \, \mathrm{d}V'\right] \mathrm{d}V \, \mathrm{d}t + \int_0^{\Delta t} \int_V \dot{\boldsymbol{y}} \bullet \left[\boldsymbol{\chi}^{\text{c}} - \frac{\partial \Psi^{\text{c}}(\xi^{\text{c}})}{\partial \xi^{\text{c}}}\right] \mathrm{d}V \, \mathrm{d}t \\ & + \int_V \boldsymbol{v} \cdot \operatorname{div} \boldsymbol{\rho} \, \mathrm{d}V - \int_{S_{\text{T}}} \boldsymbol{v} \cdot \boldsymbol{\rho} \cdot \boldsymbol{n} \, \mathrm{d}S + \int_V \boldsymbol{h} \bullet \Delta \xi \, \mathrm{d}V + \int_V \boldsymbol{r} : [\nabla^{\text{s}}(\Delta \boldsymbol{u}^{\text{r}}) - \Delta \boldsymbol{\varepsilon}^{\text{p}}] \mathrm{d}V \\ & - \int_{S_{\text{D}}} \Delta \boldsymbol{u}^{\text{r}} \cdot \boldsymbol{r} \cdot \boldsymbol{n} \, \mathrm{d}S, \end{aligned} \tag{45}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\chi}$ are defined as in (15) and (16), and $\dot{l}(\boldsymbol{x},t) \geqslant 0$, $\dot{\boldsymbol{e}}^{\text{p}}(\boldsymbol{x},t)$, $\dot{\boldsymbol{y}}(\boldsymbol{x},t)$, $\boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{h}(\boldsymbol{x})$, $\boldsymbol{r}(\boldsymbol{x})$ are the appropriate Lagrange multipliers (the meaning of which will be discovered later on).

The first variation of (45), after some mathematics (whose details are skipped for brevity), can be written in the following form:

$$\begin{aligned}\delta L_{\mathscr{K}} = & \int_0^{\Delta t}\int_V \delta\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} : \left[\frac{\partial \Phi}{\partial \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}} - \boldsymbol{\sigma}^{\mathrm{E}} - \boldsymbol{r} - \int_V \boldsymbol{Z} : \boldsymbol{e}^{\mathrm{p}}\,\mathrm{d}V'\right]\mathrm{d}V\,\mathrm{d}t + \left[\int_V \delta\boldsymbol{\varepsilon}^{\mathrm{p}} : \int_V \boldsymbol{Z} : \boldsymbol{e}^{\mathrm{p}}\,\mathrm{d}V'\,\mathrm{d}V\right]_0^{\Delta t} \\ & + \int_0^{\Delta t}\int_V \delta\dot{l} f(\boldsymbol{\sigma},\boldsymbol{\chi})\,\mathrm{d}V\,\mathrm{d}t + \int_0^{\Delta t}\int_V \delta\dot{\boldsymbol{\xi}} \bullet \left[\frac{\partial \Phi}{\partial \dot{\boldsymbol{\xi}}} + \boldsymbol{h} + \int_0^t \boldsymbol{H}(\xi) \bullet \dot{\boldsymbol{y}}\,\mathrm{d}\bar{t}\right]\mathrm{d}V\,\mathrm{d}t - \left[\delta\boldsymbol{\xi} \bullet \int_0^t \boldsymbol{H}(\xi) \bullet \dot{\boldsymbol{y}}\,\mathrm{d}\bar{t}\right]_0^{\Delta t} \\ & + \int_0^{\Delta t}\int_V \delta\boldsymbol{\sigma}^{\mathrm{rc}} : \left[-\dot{\boldsymbol{e}}^{\mathrm{p}} + \dot{l}\frac{\partial f}{\partial \boldsymbol{\sigma}}\right]\mathrm{d}V\,\mathrm{d}t + \int_0^{\Delta t}\int_V \delta\boldsymbol{\chi}^{\mathrm{c}} \bullet \left[\dot{\boldsymbol{y}} + \dot{l}\frac{\partial f}{\partial \boldsymbol{\chi}}\right]\mathrm{d}V\,\mathrm{d}t \\ & + \int_V \delta\boldsymbol{\rho} : \left[-\nabla^{\mathrm{s}}\boldsymbol{v} + \int_0^{\Delta t} \dot{l}\frac{\partial f}{\partial \boldsymbol{\sigma}}\,\mathrm{d}t\right]\mathrm{d}V + \int_{S_{\mathrm{D}}} \boldsymbol{v}\cdot\delta\boldsymbol{\rho}\cdot\boldsymbol{n}\,\mathrm{d}S - \int_V \delta(\Delta\boldsymbol{u}^{\mathrm{r}})\cdot\mathrm{div}\boldsymbol{r}\,\mathrm{d}V + \int_{S_{\mathrm{T}}} \delta(\Delta\boldsymbol{u}^{\mathrm{r}})\cdot\boldsymbol{r}\cdot\boldsymbol{n}\,\mathrm{d}S \\ & + \int_0^{\Delta t}\int_V \delta\dot{\boldsymbol{e}}^{\mathrm{p}} : \left[-\boldsymbol{\sigma}^{\mathrm{rc}} + \int_V \boldsymbol{Z} : \boldsymbol{\varepsilon}^{\mathrm{p}}\,\mathrm{d}V'\right]\mathrm{d}V\,\mathrm{d}t - \int_{S_{\mathrm{T}}} \delta\boldsymbol{v}\cdot\boldsymbol{\rho}\cdot\boldsymbol{n}\,\mathrm{d}S + \int_V \delta\boldsymbol{h} \bullet \Delta\boldsymbol{\xi}\,\mathrm{d}V \\ & + \int_V \delta\boldsymbol{r} : [\nabla^{\mathrm{s}}(\Delta\boldsymbol{u}^{\mathrm{r}}) - \Delta\boldsymbol{\varepsilon}^{\mathrm{p}}]\mathrm{d}V - \int_{S_{\mathrm{D}}} \Delta\boldsymbol{u}^{\mathrm{r}}\cdot\delta\boldsymbol{r}\cdot\boldsymbol{n}\,\mathrm{d}S. \end{aligned} \tag{46}$$

Then, in consideration that $L_{\mathscr{K}}$ takes on a minimum with respect to the primal variables, but a maximum with respect to the Lagrange multipliers, the Euler–Lagrange equations (necessary conditions, but also sufficient due to the problem convexity) associated with (41) include, besides the constraints (19)–(21) and (24)–(26), the following optimality conditions:

$$\boldsymbol{\sigma}^{\mathrm{E}} + \boldsymbol{r} + \boldsymbol{\tau}^{\mathrm{c}} = \frac{\partial \Phi}{\partial \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}}, \quad \boldsymbol{h} + \boldsymbol{X}^{\mathrm{c}} = -\frac{\partial \Phi}{\partial \dot{\boldsymbol{\xi}}} \quad \text{in } V \times (0, \Delta t), \tag{47}$$

$$\boldsymbol{\tau}^{\mathrm{c}} := \int_V \boldsymbol{Z} : \boldsymbol{e}^{\mathrm{p}}\,\mathrm{d}V, \quad \boldsymbol{X}^{\mathrm{c}} := \int_0^t \boldsymbol{H}(\xi) \bullet \dot{\boldsymbol{y}}\,\mathrm{d}\bar{t} \quad \text{in } V \times (0, \Delta t), \tag{48}$$

$$\dot{l} \geqslant 0, \quad \dot{l} f(\boldsymbol{\sigma},\boldsymbol{\chi}) = 0 \quad \text{in } V \times (0, \Delta t), \tag{49}$$

$$\dot{\boldsymbol{e}}^{\mathrm{p}} = \dot{l}\frac{\partial f(\boldsymbol{\sigma},\boldsymbol{\chi})}{\partial \boldsymbol{\sigma}}, \quad -\dot{\boldsymbol{y}} = \dot{l}\frac{\partial f(\boldsymbol{\sigma},\boldsymbol{\chi})}{\partial \xi} \quad \text{in } V \times (0, \Delta t), \tag{50}$$

$$\mathrm{div}\boldsymbol{r} = \boldsymbol{0} \text{ in } V, \quad \boldsymbol{r}\cdot\boldsymbol{n} = \boldsymbol{0} \text{ on } S_{\mathrm{T}}, \tag{51}$$

$$\Delta\boldsymbol{e}^{\mathrm{p}} := \int_0^{\Delta t} \dot{\boldsymbol{e}}^{\mathrm{p}}\,\mathrm{d}t = \nabla^{\mathrm{s}}\boldsymbol{v} \text{ in } V, \quad \boldsymbol{v} = \boldsymbol{0} \text{ on } S_{\mathrm{D}}, \tag{52}$$

$$\Delta\boldsymbol{X}^{\mathrm{c}} := \int_0^{\Delta t} \boldsymbol{H}(\xi) \bullet \dot{\boldsymbol{y}}\,\mathrm{d}t = \boldsymbol{0} \quad \text{in } V, \tag{53}$$

where $\boldsymbol{\sigma}$, $\boldsymbol{\chi}$ satisfy (15) and (16), i.e. $\boldsymbol{\sigma} := \boldsymbol{\sigma}^{\mathrm{E}} + \boldsymbol{\rho} + \boldsymbol{\sigma}^{\mathrm{rc}}$, $\boldsymbol{\chi} := \boldsymbol{q} + \boldsymbol{\chi}^{\mathrm{c}}$. The meaning of the Lagrange multipliers transpires from the above equations. In particular, one can recognize that $\boldsymbol{\tau} := \boldsymbol{r} + \boldsymbol{\tau}^{\mathrm{c}}$ is a residual stress field with $\boldsymbol{\tau}^{\mathrm{c}}$ associated with the plastic strain rates $\dot{\boldsymbol{e}}^{\mathrm{p}}$, the latter giving rise to a strain ratchet field $\Delta\boldsymbol{e}^{\mathrm{p}}$, compatible with the displacements $\boldsymbol{v}$ vanishing on S_{D}; also, $\boldsymbol{X} := \boldsymbol{h} + \boldsymbol{X}^{\mathrm{c}}$ is a static internal variable field with $\boldsymbol{X}^{\mathrm{c}}$ associated with the kinematic internal variable rate $\dot{\boldsymbol{y}}$ and such that $\Delta\boldsymbol{X} = \Delta\boldsymbol{X}^{\mathrm{c}} = \boldsymbol{0}$ in V.

By (47) one can write

$$\Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) = (\boldsymbol{\sigma}^{\mathrm{E}} + \boldsymbol{r} + \boldsymbol{\tau}^{\mathrm{c}}) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - (\boldsymbol{h} + \boldsymbol{X}^{\mathrm{c}}) \bullet \dot{\boldsymbol{\xi}} \quad \text{in } V \times (0, \Delta t). \tag{54}$$

Thus, on comparing (42) and (54) with each other, it follows that, at the optimum, it is identically: $\boldsymbol{r} = \boldsymbol{\rho}$, $\boldsymbol{h} = \boldsymbol{q}$, $\boldsymbol{\tau}^{\mathrm{c}} = \boldsymbol{\sigma}^{\mathrm{rc}}$, $\boldsymbol{X}^{\mathrm{c}} = \boldsymbol{\chi}^{\mathrm{c}}$. Also, from the latter two identities follows that $\dot{\boldsymbol{\tau}}^{\mathrm{c}} = \dot{\boldsymbol{\sigma}}^{\mathrm{rc}}$, $\dot{\boldsymbol{X}}^{\mathrm{c}} = \dot{\boldsymbol{\chi}}^{\mathrm{c}}$, that is, by (15)–(17), (48) and (50):

$$\int_V \boldsymbol{Z} : \frac{\partial f}{\partial \boldsymbol{\sigma}} (\dot{\lambda} - \dot{l}) \, \mathrm{d}V' = \mathbf{0} \quad \text{in } V \times (0, \Delta t), \tag{55}$$

$$\boldsymbol{H}(\xi) \bullet \frac{\partial f}{\partial \boldsymbol{\chi}} (\dot{\lambda} - \dot{l}) = \mathbf{0} \quad \text{in } V \times (0, \Delta t), \tag{56}$$

which imply that $\dot{\lambda} = \dot{l}$, hence $\dot{\boldsymbol{e}}^{\mathrm{p}} = \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$, $\dot{\boldsymbol{y}} = \dot{\boldsymbol{\xi}}$, identically.

The findings of here above enable one to state that the Euler–Lagrange equations of problem (41) are equivalent to the equation set (15)–(21) and (24)–(26) in the sense that the two sets of equations admit the same solution (which is unique for all, except for $\boldsymbol{\rho}$ and $\boldsymbol{q}$ within the elastic region $V_{\mathrm{e}} \subseteq V$, see Section 4). The following statement can thus be phrased:

Statement 4. A *steady cycle minimum principle* holds for a solid body, or structure, subjected to assigned cyclic (mechanical and/or kinematical) loads, which states that the structure's steady cycle makes the total net resistant work, W_{res}, take on a vanishing minimum within a suitable domain $\mathscr{K}$ of CASs, and that, conversely, the minimum solution for W_{res} in $\mathscr{K}$ identifies the steady cycle in the structure. ■

The nature of the static and kinematic parts intervening to build a generic CAS of $\mathscr{K}$, and in particular their special role played within the static (Melan) and kinematic (Koiter) shakedown theorems, respectively, suggest one to interpret the steady cycle minimum principle previously presented as a particular combined form of the two shakedown theorems, holding for loads in excess to the shakedown limit.

The above steady cycle minimum principle turns out to be an extension, to generalized standard material models, of that given by Ponter and Chen (2001) for perfect plasticity, as well as of the analogous maximum principle of Gokhfeld and Cherniavsky (1980)—although the latter authors included, among the set of constraints of the maximum problem, the redundant conditions $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\lambda} \partial f / \partial \boldsymbol{\sigma}$, $\dot{\lambda} \geqslant 0$.

7. Specializations of the steady cycle minimum principle

Two special situations are considered in this section, one of which is the case of loads not exceeding the shakedown limit, the other is the case of a material element (or specimem) subjected to a given cyclic uniform stress (or strain).

7.1. Structure under loads not exceeding the shakedown limit

For loads *below* the shakedown limit, Melan's theorem of shakedown theory asserts the existence of initial self-stress and statistical internal variable fields, say $\boldsymbol{\rho}$ and $\boldsymbol{q}$, such that $f(\boldsymbol{\sigma}^{\mathrm{E}} + \boldsymbol{\rho}, \boldsymbol{q}) < 0$ everywhere in V and for all t, $0 \leqslant t \leqslant \Delta t$. As previously noted, this implies that, for such loads, $\mathscr{K}$ contains trivial CASs, that is with $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \equiv \mathbf{0}$ and $\dot{\boldsymbol{\xi}} \equiv \mathbf{0}$. Anyone of such CASs can be recognized to provide the/an optimal solution for problem (41), which correspondingly degenerates: namely, the objective functional, as well as the related constraints, lose meaning, except the constraint (24) remaining unaltered and (19) saving its uncoupled form $f(\boldsymbol{\sigma}^{\mathrm{E}} + \boldsymbol{\rho}, \boldsymbol{q}) \leqslant 0$ in $V \times (0, \Delta t)$. In other words, problem (41) collapses into one consisting in the search for statically and plastically admissible pairs $(\boldsymbol{\rho}, \boldsymbol{q})$, in accord with the Melan shakedown theorem. All this implies that the structure's stabilized response is a purely elastic one, as expected.

For loads *at* the shakedown limit, a trivial CAS like in the previous case exists which again is the/an optimal solution for problem (41), hence the latter problem degenerates as explained before. However, the structure being in a shakedown limit state characterized by some *impending inadaptation* (*or* *noninstantaneous*) *collapse mode*, the minimum principle is expected to take on, correspondingly, a particular limit form, useful to evaluate the incipient inadaptation collapse.

In order to find out the above limit form of the minimum principle, let the elastic stress response $\boldsymbol{\sigma}^{\rm E}$ be replaced by $(1+\zeta)\boldsymbol{\sigma}^{\rm E}$, where $\zeta > 0$ is a (small) scalar. In this way, the considered load exceeds the shakedown limit (hence produces a nontrivial elastic–plastic steady cycle in the structure) for any $\zeta > 0$, but is at the shakedown limit for $\zeta = 0$. Then, with the positions:

$$\dot{\boldsymbol{\varepsilon}}_0^{\rm p}(\boldsymbol{x},t) := \frac{1}{\zeta}\dot{\boldsymbol{\varepsilon}}^{\rm p}(\boldsymbol{x},t), \quad \dot{\boldsymbol{\xi}}_0(\boldsymbol{x},t) := \frac{1}{\zeta}\dot{\boldsymbol{\xi}}(\boldsymbol{x},t) \quad \text{in } V\times(0,\Delta t), \tag{57}$$

and analogously for $\Delta\boldsymbol{u}^{\rm r}$, Eqs. (19), (25) and (26) transform, respectively, into:

$$f\big((1+\zeta)\boldsymbol{\sigma}^{\rm E} + \boldsymbol{\rho} + \boldsymbol{\sigma}^{\rm rc}, \boldsymbol{q} + \chi^{\rm c}\big) \leqslant 0 \quad \text{in } V\times(0,\Delta t), \tag{58}$$

$$\Delta\boldsymbol{\varepsilon}_0^{\rm p} := \int_0^{\Delta t} \dot{\boldsymbol{\varepsilon}}_0^{\rm p}\,{\rm d}t = \nabla^{\rm s}(\Delta\boldsymbol{u}_0^{\rm r}) \text{ in } V, \quad \Delta\boldsymbol{u}_0^{\rm r} = \boldsymbol{0} \text{ on } S_{\rm D}, \tag{59}$$

$$\Delta\boldsymbol{\xi}_0 := \int_0^{\Delta t} \dot{\boldsymbol{\xi}}_0\,{\rm d}t = \boldsymbol{0} \quad \text{in } V, \tag{60}$$

all of which hold for $\zeta > 0$.

The *reduced* equation set in now constituted by Eqs. (20), (21), (24) and (58)–(60), whereas the CAS domain $\mathscr{K}$ of (34) correspondingly takes on the form:

$$\mathscr{K}_0 := \left\{ \begin{array}{l} \dot{\boldsymbol{\varepsilon}}_0^{\rm p}(\boldsymbol{x},t), \dot{\boldsymbol{\xi}}_0(\boldsymbol{x},t), \boldsymbol{\rho}(\boldsymbol{x}), \boldsymbol{q}(\boldsymbol{x}), \boldsymbol{x}\in V, 0\leqslant t\leqslant \Delta t : \\ \text{s.t. constraints} \quad (20), (21), (24), (58)\text{–}(60) \end{array} \right\}. \tag{61}$$

Accordingly, problem (41) can be rewritten in the following equivalent form:

$$\min_{\mathscr{K}_0} W_{\rm res(0)} := \int_0^{\Delta t}\int_V \left[\Phi(\dot{\boldsymbol{\varepsilon}}_0^{\rm p}, \dot{\boldsymbol{\xi}}_0) - (1+\zeta)\boldsymbol{\sigma}^{\rm E} : \dot{\boldsymbol{\varepsilon}}_0^{\rm p}\right]{\rm d}V\,{\rm d}t \tag{62}$$

where, in analogy to (57), $W_{\rm res(0)} = W_{\rm res}/\zeta$.

For any $\zeta > 0$, the latter problem is meaningful and in fact equivalent to (41) written with $\boldsymbol{\sigma}^{\rm E}$ replaced by $(1+\zeta)\boldsymbol{\sigma}^{\rm E}$; however, for $\zeta \to 0$, whereas the latter problem (41) loses meaning, on the contrary problem (62) does not. In fact, one can observe: (i) at the limit for $\zeta \to 0$, the constraints (20) and (21) disappear since no plastic strains occur at the shakedown limit, that is, $\dot{\boldsymbol{\varepsilon}}^{\rm p} \to \boldsymbol{0}$, $\dot{\boldsymbol{\xi}} \to \boldsymbol{0}$, hence $\boldsymbol{\sigma}^{\rm rc} \to \boldsymbol{0}$, $\boldsymbol{\chi}^{\rm c} \to \boldsymbol{0}$, and thus (58) takes on the (uncoupled) limit form

$$f(\boldsymbol{\sigma}^{\rm E} + \boldsymbol{\rho}, \boldsymbol{q}) \leqslant 0 \quad \text{in } V\times(0,\Delta t); \tag{63}$$

(ii) since the ratios in (57) remain meaningful even for $\zeta \to 0$, Eqs. (59) and (60) save their forms also in the limit for $\zeta \to 0$; and finally, (iii) the constraint (24) remains unchanged. It follows that the CAS domain $\mathscr{K}_0$ of (61) is meaningful also in the limit for $\zeta \to 0$, but with the constraint (58) replaced by (63). Consequently, one can recognize that problem (62) decouples into two subproblems, one of which is a minimum problem of kinematic nature, that is:

$$\left. \begin{array}{l} \min\limits_{(\dot{\boldsymbol{\varepsilon}}_0^{\rm p}, \dot{\boldsymbol{\xi}}_0)} W_{\rm res(0)} := \int_0^{\Delta t}\int_V \left[\Phi(\dot{\boldsymbol{\varepsilon}}_0^{\rm p}, \dot{\boldsymbol{\xi}}_0) - \boldsymbol{\sigma}^{\rm E} : \dot{\boldsymbol{\varepsilon}}_0^{\rm p}\right]{\rm d}V\,{\rm d}t \\ \text{s.t. constraints (59) and (60)} \end{array} \right\}, \tag{64}$$

whereas the other is a search problem of static nature governed by Eqs. (24) and (63). The latter problem consists in the search for a statically admissible initial stress/hardening state satisfying (63) and is thus recognized to comply with the Melan theorem of shakedown theory (Halphen, 1979; Comi and Corigliano, 1991; Polizzotto et al., 1991).

Problem (64) is formally coincident with the Marcovian principle of shakedown theory (De Saxcé, 1995), but extended to generalized standard materials (Polizzotto et al., 2000). So interpreted, it is valid for any load level, since in fact one can observe:

- For loads at the shakedown limit, problem (64) admits a vanishing minimum and the related optimal solution $(\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{p}}, \dot{\boldsymbol{\xi}}_0)$ describes the structure's impending inadaptation collapse mode (or incipient steady cycle);
- For loads below the shakedown limit, problem (64) admits the trivial solution $\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{p}} = \mathbf{0}$, $\dot{\boldsymbol{\xi}}_0 = \mathbf{0}$;
- For loads exceeding the shakedown limit, problem (64) admits no minimum.

These features of problem (64) can be established either by considering it as the limit of problem (61) for $\zeta \to 0$, or by studying the related Euler–Lagrange equations, but the details of this issue are skipped for brevity sake. Problem (64) is therefore equivalent to the Koiter (1960) kinematic theorem of shakedown theory (Halphen, 1979; Comi and Corigliano, 1991; Polizzotto et al., 1991, 2000).

It can be concluded that the steady cycle minimum principle of Section 6—already qualified as a special combination of the static and kinematic shakedown theorems holding for loads in excess to the shakedown limit—decouples into the two distinct shakedown statements for loads not exceeding the shakedown limit.

It is worth noting that problem (64) can be viewed as a particular time/space integrated form of the minimum net resistant power theorem of Section 3. As such, it solves the problem to find the (incipient) steady cycle for a structure being subjected to loads not exceeding the shakedown limit.

7.2. Specimen subjected to uniform cyclic stress (or strain)

Let a material element (or specimen) be subjected to a (uniform) cyclic stress, say $\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}(t), 0 \leqslant t \leqslant \Delta t$. In analogy to a structure subjected to a cyclic load, a steady cycle can be determined for the material element (Lemaitre and Chaboche, 1990). Eqs. (15)–(21) and (24)–(26) simplify as in the following (no stress equilibrium, nor strain compatibility must be considered):

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\lambda}\frac{\partial f}{\partial \bar{\boldsymbol{\sigma}}}, \quad -\dot{\xi} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\chi}} \quad \text{in } (0, \Delta t), \tag{65}$$

$$\dot{\lambda} \geqslant 0, \quad \dot{\lambda} f = 0 \quad \text{in } (0, \Delta t), \tag{66}$$

$$f = f(\bar{\boldsymbol{\sigma}}, \boldsymbol{q} + \boldsymbol{\chi}^{\mathrm{c}}) \leqslant 0 \quad \text{in } (0, \Delta t), \tag{67}$$

$$\boldsymbol{\chi}^{\mathrm{c}} = \boldsymbol{\chi}^{\mathrm{c}}(\xi^{\mathrm{c}}) := \frac{\partial \Psi^{\mathrm{c}}(\xi^{\mathrm{c}})}{\partial \xi^{\mathrm{c}}} \quad \text{in } (0, \Delta t), \tag{68}$$

$$\Delta\boldsymbol{\xi} := \int_0^{\Delta t} \dot{\boldsymbol{\xi}}\, \mathrm{d}t = \mathbf{0}, \tag{69}$$

where $\boldsymbol{\chi} = \boldsymbol{q} + \boldsymbol{\chi}^{\mathrm{c}}$, $\xi = \boldsymbol{p} + \xi^{\mathrm{c}}$, $\boldsymbol{q} = \boldsymbol{\chi}(\boldsymbol{p})$, hence $\dot{\boldsymbol{\xi}} = \dot{\boldsymbol{\xi}}^{\mathrm{c}}$, $\Delta\xi = \Delta\xi^{\mathrm{c}}$, and moreover $\Psi^{\mathrm{c}}(\xi^{\mathrm{c}})$ is given by (22). The (unique) solution to Eqs. (65)–(69) describes the specimen's steady cycle.

The above equation set admits a variational formulation through a minimum principle which is a particular case of that of Section 6. In order to show this point, a procedure similar to that adopted in Section 6.1 is used. The above equation set is relaxed by disregarding Eqs. (65) and (66), so obtaining a *reduced* equation set, i.e. Eqs. (67)–(69), such that the pertinent CAS domain reads:

$$\mathscr{K} := \{\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}(t), \dot{\boldsymbol{\xi}}(t), \boldsymbol{q}, 0 \leqslant t \leqslant \Delta t : \text{s.t. constraints (67)–(69)}\}. \tag{70}$$

Any CAS produces a total static internal variable, $\boldsymbol{\chi} = \boldsymbol{q} + \boldsymbol{\chi}^{\mathrm{c}}$, periodic like the applied stress ($\boldsymbol{\chi}^{\mathrm{c}} = \mathbf{0}$ at $t = 0$ and $t = \Delta t$) and such that the pair $(\bar{\boldsymbol{\sigma}}, \boldsymbol{\chi})$ is plastically admissible at all times in the cycle. There certainly exists in $\mathscr{K}$ a particular CAS which, together with some $\dot{\lambda}(t)$, $0 \leqslant t \leqslant \Delta t$, solve the complete equation set (65)–(69).

Then, let a generic CAS—with the related static internal variable increment (68)—be used to construct the functional

$$W_{\mathrm{res}} := \int_0^{\Delta t} \left[\Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) - \bar{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + (\boldsymbol{q} + \boldsymbol{\chi}^{\mathrm{c}}) \bullet \dot{\boldsymbol{\xi}}\right] \mathrm{d}t \geqslant 0 \quad \text{in } \mathscr{K}, \tag{71}$$

which is a time integrated form of the net resistant power, $w_{\mathrm{res}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$, of Section 3, adapted to the present context. W_{res} turns out to be nonnegative for whatever CAS due to (6) and (67), or equivalently to (67) and the minimum net resistant power theorem—the degenerate case being automatically excluded by (67)—. Because of (68) and (69), and remembering the identities $\dot{\boldsymbol{\xi}} = \dot{\boldsymbol{\xi}}^{\mathrm{c}}$ and $\Delta\boldsymbol{\xi} = \Delta\boldsymbol{\xi}^{\mathrm{c}}$, one has

$$\int_0^{\Delta t} (\boldsymbol{q} + \boldsymbol{\chi}^{\mathrm{c}}) \bullet \dot{\boldsymbol{\xi}} \, \mathrm{d}t = \boldsymbol{q} \bullet \Delta\boldsymbol{\xi}^{\mathrm{c}} + \Psi^{\mathrm{c}}(\Delta\boldsymbol{\xi}^{\mathrm{c}}) - \Psi^{\mathrm{c}}(\mathbf{0}) = 0, \tag{72}$$

in virtue of which W_{res} simplifies into

$$W_{\mathrm{res}} = \int_0^{\Delta t} \left[\Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) - \bar{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\right] \mathrm{d}t \geqslant 0 \quad \text{in } \mathscr{K}. \tag{73}$$

The latter expression shows that W_{res} has the meaning of *total net resistant work*, which equals the difference between the total intrinsic dissipation energy and the work correspondingly done by the applied load. Since the steady state solution, that is the solution to (65)–(69), is formed with a particular CAS, and since $W_{\mathrm{res}} = 0$ if (73) is computed using this particular CAS, it follows that the functional W_{res} admits a vanishing minimum in $\mathscr{K}$, and also that the minimum solution characterizes the steady cycle.

Thus, considering the minimum problem

$$\min_{\mathscr{K}} W_{\mathrm{res}} := \int_0^{\Delta t} \left[\Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) - \bar{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\right] \mathrm{d}t \tag{74}$$

and observing that $W_{\mathrm{res}} = 0$ at the minimum, one has that the nonnegative square-bracketed integrand of (71) must also vanish, hence

$$\Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) = \bar{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - (\boldsymbol{q} + \boldsymbol{\chi}^{\mathrm{c}}) \bullet \dot{\boldsymbol{\xi}} \quad \forall t \in (0, \Delta t). \tag{75}$$

It follows that, at the optimum, it is

$$\bar{\boldsymbol{\sigma}} = \frac{\partial \Phi}{\partial \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}}, \quad \boldsymbol{q} + \boldsymbol{\chi}^{\mathrm{c}} = -\frac{\partial \Phi}{\partial \dot{\boldsymbol{\xi}}} \quad \forall t \in (0, \Delta t), \tag{76}$$

that is, the optimal pairs $(\bar{\boldsymbol{\sigma}}, \boldsymbol{\chi} := \boldsymbol{q} + \boldsymbol{\chi}^{\mathrm{c}})$ and $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ are mutually related by the plasticity laws (65)–(67). Therefore, the optimal CAS with some $\dot{\lambda}(t)$ solve the complete equation set (65)–(69).

Similarly to the general case of Section 6, the mechanical implications of problem (74) can be discovered by studying the relevant optimality equations through the classical Lagrange multiplier method, but this point is not further pursued because the procedure to adopt is quite similar to that employed in Section 6.2.

It can be concluded that the steady cycle minimum principle for the specimen under uniform cyclic stress coincides with that obtainable from the general principle of Section 6 with the appropriate particularizations, that is, by eliminating the self-stresses $\boldsymbol{\rho}$ (together with the related equilibrium conditions) and the strain compatibility equations, as well as the dependence upon the space co-ordinates $\boldsymbol{x}$.

In the case of imposed cyclic strain, say $\bar{\boldsymbol{\varepsilon}}(t)$, on the material element, the steady cycle can be found by a procedure similar to that presented in this section, but a few changes are needed. In fact, the stress $\boldsymbol{\sigma}$ is now an additional unknown, whereas the equations $\boldsymbol{\sigma} = \boldsymbol{D} : (\bar{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}^{\mathrm{p}})\ \forall t \in (0, \Delta t)$ and $\Delta\boldsymbol{\varepsilon}^{\mathrm{p}} = \mathbf{0}$ are additional constraints. Moreover, the total net resistant work has now the expression:

$$W_{\mathrm{res}} = \int_0^{\Delta t} \left[\Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) - \bar{\boldsymbol{\varepsilon}} : \boldsymbol{D} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\right] \mathrm{d}t. \tag{77}$$

This case is not further pursued here for sake of brevity.

8. Criteria for the ratchet limit loads

For practical design purposes, it is paramount to distinguish the cyclic loads which cause the structure to exhibit ratchetting in the long-term response (with consequent plastic strain growth) from those under which the ratchetting phenomenon is escaped (and thus the structure can only experience either alternating plasticity, or shakedown for lower load values). The steady cycle minimum principle of Section 6 can be utilized to derive a criterion for recognizing whether a given cyclic load exceeds, or not, the *ratchet limit*, that is, whether it produces, or not, ratchetting in the long-term response. This task is addressed in the present section, first for general loads, then for loads not exceeding the shakedown limit.

8.1. General

Let the compatibility constraint (25), concurring to qualify the CASs of $\mathscr{K}$ in (34), be replaced by the more stringent one stating that the plastic strain ratchet is identically vanishing, that is

$$\Delta\boldsymbol{\varepsilon}^{\mathrm{p}} := \int_0^{\Delta t} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\, \mathrm{d}t = \mathbf{0} \quad \text{in } V. \tag{78}$$

A subdomain of *no-ratchet* CASs, say $\widetilde{\mathscr{K}} \subset \mathscr{K}$, is obtained by restricting the kinematic parts $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}})$ to those obeying (78), that is:

$$\widetilde{\mathscr{K}} := \left\{ \begin{array}{l} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}(\boldsymbol{x}, t), \dot{\boldsymbol{\xi}}(\boldsymbol{x}, t), \boldsymbol{\rho}(\boldsymbol{x}), \boldsymbol{q}(\boldsymbol{x}), \boldsymbol{x} \in V, 0 \leqslant t \leqslant \Delta t : \\ \text{s.t. constraint (19)–(21), (24), (26), (78)} \end{array} \right\}. \tag{79}$$

Then, using a generic CAS of (79), a functional W_{res}, formally equal to that of (35) and (41), can be generated, that is

$$W_{\mathrm{res}} := \int_0^{\Delta t} \int_V \left[\Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}) - \boldsymbol{\sigma}^{\mathrm{E}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\right] \mathrm{d}V\, \mathrm{d}t \quad \text{in } \widetilde{\mathscr{K}}. \tag{80}$$

Obviously, this $W_{\mathrm{res}} = W_{\mathrm{res}}[\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\xi}}, \boldsymbol{\rho}, \boldsymbol{q}]$ turns out to be nonnegative for all no-ratchet CAS. It can be easily recognized that, if the given load *does not* exceed the ratchet limit, there actually exists a particular CAS in $\widetilde{\mathscr{K}}$ for which $W_{\mathrm{res}} = 0$, and that, if on the contrary the given load *does* exceed the ratchet limit, necessarily $W_{\mathrm{res}} > 0$ for all CASs in $\widetilde{\mathscr{K}}$. The following statement can thus be given:

Statement 5. For a cyclically loaded structure, a *criterion for ratchet limit loads* holds, which states that a given load does not exceed the ratchet limit if there exists a no-ratchet CAS for which the (nonnegative) total net resistant work, W_{res}, is vanishing, whereas the given load does exceed the ratchet limit if $W_{\mathrm{res}} > 0$ for every no-ratchet CAS. ■

The above criterion can be set in an alternative more stringent form by invoking the following minimum problem, analogous to (41):

$$\min_{\widetilde{\mathscr{K}}} W_{\text{res}} := \int_0^{\Delta t} \int_V \left[\Phi(\dot{\boldsymbol{\varepsilon}}^{\text{p}}, \dot{\boldsymbol{\xi}}) - \boldsymbol{\sigma}^{\text{E}} : \dot{\boldsymbol{\varepsilon}}^{\text{p}} \right] \text{d}V\,\text{d}t. \tag{81}$$

This problem certainly admits a minimum, the value of which is zero if, and only if, the given load does not exceed the ratchet limit, but positive in the contrary case. The following can thus be stated:

Statement 6. For a cyclically loaded structure, a (*stringent*) *criterion for ratchet limit loads* holds, which states that a given load does not exceed the ratchet limit if, and only if, $\min W_{\text{res}}$ in $\widetilde{\mathscr{K}}$ is zero, whereas the given load does exceed the ratchet limit if $\min W_{\text{res}}$ in $\widetilde{\mathscr{K}}$ is positive. ■

Note that, if $\widetilde{\mathscr{K}}$ contains a trivial CAS, that is one of the type $(\dot{\boldsymbol{\varepsilon}}^{\text{p}} \equiv \mathbf{0}, \dot{\boldsymbol{\xi}} \equiv \mathbf{0}, \boldsymbol{\rho}, \boldsymbol{q})$ for which $W_{\text{res}} = 0$, hence $\min W_{\text{res}}$ in $\widetilde{\mathscr{K}}$ is zero, then the given cyclic load does not exceed the shakedown limit, what is in accord with the Melan theorem of shakedown theory.

Also note that the Lagrangian functional associated with (81)—not reported here for brevity—can be easily shown to be like (45), but without all integral terms containing $\Delta\boldsymbol{u}^{\text{r}}$, and thus in the present case Eq. (51) does not hold (i.e. $\boldsymbol{r}$ is not necessarily a self-stress). However, provided that the minimum of (81) is vanishing, Eq. (54) and all subsequent arguments do hold also in the present case, till the conclusion that $\boldsymbol{r} = \boldsymbol{\rho}$, $\boldsymbol{h} = \boldsymbol{q}$, $\boldsymbol{\tau}^{\text{c}} = \boldsymbol{\sigma}^{\text{rc}}$, $\boldsymbol{X}^{\text{c}} = \boldsymbol{\chi}^{\text{c}}$, hence $\dot{\boldsymbol{e}}^{\text{p}} = \dot{\boldsymbol{\varepsilon}}^{\text{p}}$, $\dot{\boldsymbol{y}} = \dot{\boldsymbol{\xi}}$, all identically.

8.2. Loads not exceeding the shakedown limit

For loads not exceeding the shakedown limit, an appropriate form of the ratchetting limit load criterion can be devised making use of a suitable limit form of (79) and (81). This limit form can be obtained by a procedure like that employed in Section 7.1 to derive (64) from (34) and (41). The result of this procedure coincides with that previously obtained in Section 7.1, but problem (64) is to be replaced by the following:

$$\left.\begin{array}{l} \min\limits_{(\dot{\boldsymbol{\varepsilon}}_0^{\text{p}}, \dot{\boldsymbol{\xi}}_0)} W_{\text{res}(0)} := \int_0^{\Delta t} \int_V \left[\Phi(\dot{\boldsymbol{\varepsilon}}_0^{\text{p}}, \dot{\boldsymbol{\xi}}_0) - \boldsymbol{\sigma}^{\text{E}} : \dot{\boldsymbol{\varepsilon}}_0^{\text{p}} \right] \text{d}V\,\text{d}t \\ \text{s.t. } \Delta\boldsymbol{\varepsilon}_0^{\text{p}} = \mathbf{0},\ \Delta\boldsymbol{\xi}_0 = \mathbf{0} \text{ in } V \end{array}\right\}, \tag{82}$$

where the constraints (60) and (78) have been reported in abridged form.

Then, remembering the features of problem (64), the following can be stated:

Statement 7. For a structure subjected to cyclic loads not exceeding the shakedown limit, a (*stringent*) *criterion for ratchet limit loads* holds: it states that a given load does not exceed the ratchet limit (hence is either a shakedown load, or one at the alternating plasticity limit) if, and only if, problem (82) has a vanishing minimum, whereas it is at the ratchet limit if problem (82) has a positive minimum. Otherwise, if problem (82) has no minimum—this is the case when there exists some $(\dot{\boldsymbol{\varepsilon}}_0^{\text{p}}, \dot{\boldsymbol{\xi}}_0)$ such that $W_{\text{res}(0)} < 0$—, then the given load exceeds the shakedown limit. ■

The above criterion recalls an analogous criterion given by König (1979) for loads at the alternating plasticity limit.

9. The ratchet limit load problem

The evaluation of the ratchet limit loads is an important research issue for design purposes because in many instances structures, or parts of them, are proportioned against ratchetting collapse modes, while alternating plastic strains are tolerated—but with due care for their effects on the structure's working life length.

9.1. Preliminary arguments and review of known results

For the present purposes, it is useful to mention (Polizzotto, 1993a, 1994a,b) that any cyclic load, say $P(t)$, which exceeds the ratchet limit, can always be transformed into one not exceeding this limit by simply superposing to it a permanent (mechanical) load, say P^0. This means that the combined load $P^0 \cup P(t)$ (with P^0 suitably chosen) will induce either alternating plasticity for higher values of the cyclic load, or shakedown for lower values of the same load. The cyclic load value that separates the latter two types of steady state response from each other is the so-called *alternating plasticity* (or *plastic shakedown*) *limit*, (Polizzotto, 1993c).

The superposed load P^0 can in principle be constituted by arbitrarily distributed body forces in V and surface forces on S_T. However, often the loading program consists of combined loads of the type $\alpha\overline{P}^0 \cup \beta\overline{P}(t)$, where $\overline{P}^0$ and $\overline{P}(t)$ are specified reference loads and α, β are scalar multipliers. In the latter case, there is an obvious convenience in choosing the superposed load within the family $\alpha\overline{P}^0$.

Fig. 1 is a schematic representation of the so-called interaction (or Bree-like) diagram in the (α, β)-plane. In this, the region (say B) collecting all loads not exceeding the plastic collapse limit, can be divided into zones, i.e. $B = B_S \cup B_F \cup B_R$, where B_S collects the shakedown loads (including those below the elastic limit), B_F the alternating plasticity loads, and B_R the ratchetting loads, see e.g. Gokhfeld and Cherniavsky (1980), Ponter (1983), Polizzotto (1993a,b, 1994a,b). The line $\beta = \beta_{al}$ (b–d in Fig. 1) corresponds to the alternating plasticity limit. The zone $B_{NR} := B_S \cup B_F$ (dashed in Fig. 1) is the *no ratchetting zone*, whereas its boundary line ($a - b - c - d - e$ in Fig. 1) collects the loads at the *ratchet limit*. Any load in the B_R zone, as P_1 and P_2 in Fig. 1, can be transformed, by superposition of a suitable permanent load $\alpha\overline{P}^0$, into one belonging to B_{NR}, i.e. to B_S for $\beta < \beta_{al}$, or to B_F for $\beta > \beta_{al}$.

As known from the literature (Polizzotto, 1993a, 1994a,b), a structure which finds itself in a condition of alternating plasticity under a cyclic load $P(t)$, experiences alternating plastic strains only within a part of its

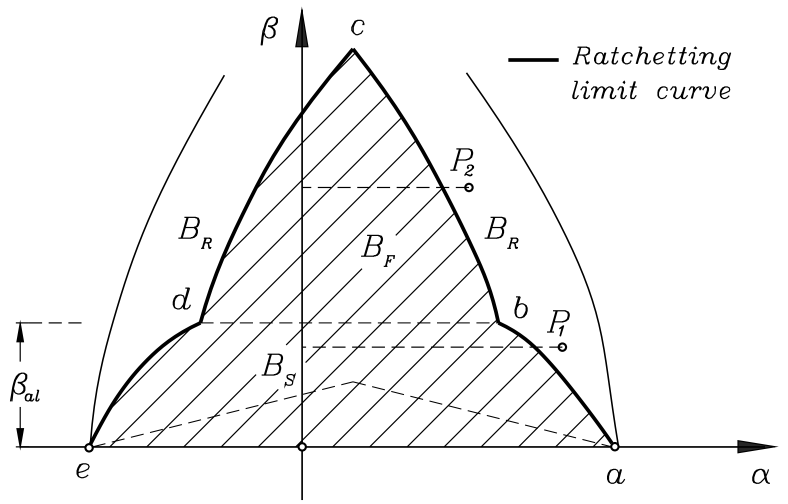

Fig. 1. Geometrical sketch of a typical Bree-like interaction diagram for a two-parameter cyclic/steady load, $\alpha\overline{P}^0 \cup \beta\overline{P}(t), 0 \leqslant t \leqslant \Delta t$: $B_S \to$ shakedown zone, $B_F \to$ alternating plasticity zone, $B_R \to$ ratchetting zone, $B_{NR} = B_S \cup B_F \to$ no ratchetting zone (dashed area), $\beta_{al} \to$ alternating plasticity limit.

volume, say $V_p \subset V$, whereas in the complementary region V_e no plastic strains occur in the steady state. Additionally, this alternating plasticity condition turns out to be *insensitive* to the application of superposed permanent loads $P^0 = \alpha\overline{P}^0$ upon V_e, in the sense that the combined load $\alpha\overline{P}^0 \cup P(t)$ does promote the same steady-state strain cycle, provided α is within some *no ratchetting range*, say $\alpha^- < \alpha < \alpha^+$, where the limits α^- and α^+ mark the no ratchetting zone boundary and are referred to as the *ratchet limits* after Ponter and Chen (2001).

For cyclic loads $P(t) = \beta\overline{P}(t)$ such that $\beta < \beta_{al}$ (e.g. P_1 in Fig. 1), the determination of α^+ and α^- can be accomplished by solving a classical shakedown limit load problem. For instance, for α^+ one has:

$$\alpha^+ = \max_{\alpha,s} \alpha \quad \text{s.t.} \begin{cases} f(\boldsymbol{\sigma}^E + \boldsymbol{s}, \boldsymbol{q}) \leqslant 0 \quad \text{in } V \times (0, \Delta t) \\ \text{Equilibrium conds. on } \boldsymbol{s} \text{ with } \alpha\overline{P}^0. \end{cases} \tag{83}$$

In fact, this problem (as well as the analogous one, $\alpha^- = \min \alpha$ under the same constraints) solves the problem of what permanent load must be superposed to $P(t)$ such that the combined load is a shakedown limit load.

For cyclic loads $P(t) = \beta\overline{P}(t)$ with $\beta > \beta_{al}$, a procedure to evaluate the ratchet limits α^+ and α^- is presented in the next subsection.

9.2. Procedure to evaluate the ratchet limits

This procedure is based on two ingredients, that is, the (stringent) ratchet limit load criterion (Statement 6) and the insensitivity features of the alternating plasticity state mentioned previously.

Let $P(t)$ be an alternating plasticity cyclic load, to be used to generate the alternating plasticity strain cycle associated with the load family $P(t) \cup \alpha\overline{P}^0$, $\alpha^- < \alpha < \alpha^+$, of which the considered $P(t)$ is, by definition, the *master load*. The ratchet limit load criterion of Statement 6 asserts that the minimum problem (81), solved for the load in question, gives

$$\min_{\widetilde{\mathscr{K}}} W_{res}\left[\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}}, \boldsymbol{\rho}, \boldsymbol{q}\right] \to 0 \tag{84}$$

with a *nontrivial* optimal CAS, say $(\dot{\boldsymbol{\varepsilon}}^{p*}, \dot{\boldsymbol{\xi}}^*, \boldsymbol{\rho}^*, \boldsymbol{q}^*) \in \widetilde{\mathscr{K}}$. The latter CAS turns out to be unique for the alternating plastic strain rates $\dot{\boldsymbol{\varepsilon}}^{p*}$ and the related kinematic internal variable rates $\dot{\boldsymbol{\xi}}^*$, together with the region $V_p \subset V$ where they take place, as well as for the self-stress, $\boldsymbol{\rho}^*$, and static internal variables, $\boldsymbol{q}^*$, but these $\boldsymbol{\rho}^*$ and $\boldsymbol{q}^*$ being however indeterminate within the elastic region $V_e = V \setminus V_p$ (where no plastic strains occur in the steady state). Obviously, the above (starred) CAS complies with all the constraints qualifying the CASs of $\widetilde{\mathscr{K}}$ in (79), i.e. the constraints (19)–(21), (24), (26) and (78), as long with the optimality conditions (15)–(18). In particular, it is:

$$f(\boldsymbol{\sigma}^E + \boldsymbol{\rho}^* + \boldsymbol{\sigma}^{rc*}, \boldsymbol{q}^* + \boldsymbol{\chi}^{c*}) \leqslant 0 \quad \text{in } V \times (0, \Delta t), \tag{85}$$

where the equality sign holds (i.e. the yield limit is actually attained) for at least two times everywhere in V_p, but nowhere in V_e.

By the insensitivity features of the alternating plasticity state—the latter being described by the starred CAS previously obtained—, permanent loads as $P^0 = \alpha\overline{P}^0$ can be superposed upon the (nonempty) elastic region V_e of the body without causing a change in the steady cycle, except for some stress increments, say $\boldsymbol{s} = \boldsymbol{s}(\boldsymbol{x})$, appearing within V_e such as to equilibrate the load $\alpha\overline{P}^0$ there superposed (Ponter and Karadenitz, 1985; Polizzotto, 1993a, 1994a,b). The set of α values for which the above insensitivity features persist fill the no ratchetting range, $\alpha^- \leqslant \alpha \leqslant \alpha^+$, and thus the ratchet limits identify with the extremes of this interval. So, with the positions:

$$\boldsymbol{\sigma}^* := \boldsymbol{\sigma}^{\mathrm{E}} + \boldsymbol{\rho}^* + \boldsymbol{\sigma}^{\mathrm{rc}*}, \quad \boldsymbol{\chi}^* := \boldsymbol{q}^* + \boldsymbol{\chi}^{\mathrm{c}*} \quad \text{in } V \times (0, \Delta t), \tag{86}$$

one can write, for example for α^+:

$$\alpha^+ = \max_{\alpha, \boldsymbol{s}} \alpha \quad \text{s.t.} \begin{cases} f(\boldsymbol{\sigma}^* + \boldsymbol{s}, \boldsymbol{\chi}^*) \leqslant 0 \text{ in } V \times (0, \Delta t) \\ \text{Equilibrium conditions on } \boldsymbol{s} \text{ with } \alpha \overline{P}^0 \end{cases} \tag{87}$$

and analogously for $\alpha^- = \min \alpha$ under the same constraints.

Note that, due to (85), nonzero values of $\boldsymbol{s}$ are allowed only within the elastic region V_{e}, where the permanent load is applied, hence $\boldsymbol{s} = \mathbf{0}$ in V_{p}. Problem (87) has the form of a classical shakedown limit load problem and it in fact is similar to (83). However, in (83) the elastic stress response, $\boldsymbol{\sigma}^{\mathrm{E}}$, to the given cyclic load operates and the yield condition is enforced with respect to some subsequent yield surface at a fixed hardening state, $\boldsymbol{q}$; whereas in (87) the stress $\boldsymbol{\sigma}^*$ and the static internal variable $\boldsymbol{q}^*$ derived from (84) (with $\boldsymbol{\rho}^*$ and $\boldsymbol{q}^*$ suitably continued in V_{e}) intervene and the yield condition is enforced with respect to an accordingly varying subsequent yield surface.

Problem (87) is the extension to the present context of the analogous problem formulated by Ponter and Chen (2001) for perfect plasticity; here in addition the conceptual background on which the problem formulation is rooted has been provided. Ponter and Karadenitz (1985) proposed approximate procedures to evaluate the (starred) alternating plasticity response needed in (87) for the perfect plasticity case; these procedures were improved (and rendered exact for one-dimensional structures) by Polizzotto (1993a, 1994a,b).

Since in (87) $\boldsymbol{s} = \mathbf{0}$ in V_{p} and thus the inequality constraint can there be enforced only in V_{e} (where $\alpha \overline{P}^0$ is applied, hence $\boldsymbol{s}$ cannot vanish), follows that problem (87) can be interpreted as a classical shakedown limit load problem for the portion V_{e} of the body, which is left after removal of V_{p}. This interpretation was named *partial shakedown* by Ponter and Karadenitz (1985) and investigated by Polizzotto (1993a, 1994a,b).

9.3. The alternating plasticity master load

An aspect of this theory not pointed out by Ponter and Chen (2001) is that, for the applicability of the procedure of the preceding subsection, the considered cyclic load $P(t)$ is required to be an alternating plasticity load. Often in practice the given cyclic load is one of this sort, especially when it is a kinematic load (e.g. thermal load), and anyway the criteria of Section 8 can be used to recognize whether this is true. The question posed here is: how an alternating plasticity load, to be used as a master load in the ratchet limit analysis, can be derived from a given $P(t)$ exceeding this limit? Again, the ratchet limit load criterion of Statement 6 together with the insensitivity features of the alternating plasticity state can be usefully employed for this purpose.

Let $P(t) = \beta \overline{P}(t)$, $\beta > \beta_{\mathrm{al}}$, be the given cyclic load, by hypothesis exceeding the ratchet limit. According to the criterion of Statement 6, the minimum problem (81), solved for the load in question, gives a *positive* optimal value of the objective functional.

A permanent load in the family $\alpha \overline{P}^0$ can always be found such that the combined load, $P(t) \cup \alpha \overline{P}^0$, does not exceed the ratchet limit. The latter condition can be enforced by the same criterion mentioned above, provided that the CAS domain (79) is suitably widened by replacing inequality (19) with the following one:

$$f(\boldsymbol{\sigma}^{\mathrm{E}} + \alpha \bar{\boldsymbol{\sigma}}^0 + \boldsymbol{\rho} + \boldsymbol{\sigma}^{\mathrm{rc}}, \boldsymbol{q} + \boldsymbol{\chi}^{\mathrm{c}}) \leqslant 0 \quad \text{in } V \times (0, \Delta t) \tag{88}$$

where $\alpha \bar{\boldsymbol{\sigma}}^0$ is the elastic stress response to $\alpha \overline{P}^0$. For every fixed value of α, a no-ratchet CAS domain, say $\widetilde{\mathscr{K}}_\alpha$, is generated, i.e.

$$\widetilde{\mathscr{K}}_\alpha := \begin{Bmatrix} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}(\boldsymbol{x}, t), \dot{\boldsymbol{\xi}}(\boldsymbol{x}, t), \boldsymbol{\rho}(\boldsymbol{x}), \boldsymbol{q}(\boldsymbol{x}), \boldsymbol{x} \in V, 0 \leqslant t \leqslant \Delta t) : \\ \text{s.t. constraints} \quad (20), (21), (24), (26), (78), (88) \end{Bmatrix}. \tag{89}$$

It follows that any *fixed* α value such that

$$\min_{\widetilde{\mathscr{K}_\alpha}} W_{\rm res}\left[\dot{\boldsymbol{\epsilon}}^{\rm p}, \dot{\boldsymbol{\xi}}, \boldsymbol{\rho}, \boldsymbol{q}\right] \to 0 \tag{90}$$

can be utilized to obtain the needed master load as the load $P(t) \cup \alpha \overline{P}^0$.

Note that one can write, formally:

$$\left.\begin{array}{l} \alpha^+ = \max \alpha \\ \alpha^- = \min \alpha \end{array}\right\} \quad \text{s.t.} \ \min_{\widetilde{\mathscr{K}_\alpha}} W_{\rm res}\left[\dot{\boldsymbol{\varepsilon}}^{\rm p}, \dot{\boldsymbol{\xi}}, \boldsymbol{\rho}, \boldsymbol{q}\right] \to 0. \tag{91}$$

The maximum operation in (87), as well as the associated minimum one, are analytical implementations of the analogous operations in (91), respectively.

10. Comments and conclusion

For a solid (or structure) composed of generalized standard material (that is, elastic–plastic material with internal variables and convex hardening potential) exhibiting hardening saturation capability, and subjected to cyclic loads, the equation set governing the long-term stabilized (or steady state) response has been discussed together with the related uniqueness features. This response has been regarded as that of a structural system subjected to a given loading cycle and being in an appropriate initial stress and hardening state. The well-known fact that this initial stress/hardening state coincides with that for which the cycle increment of plastic strains (plastic strain ratchet) is self-compatible and the analogous increment of kinematic internal variables is identically vanishing, has been here rediscovered through a maximum principle which discerns, between the set of all possible initial conditions, those for which the maximum is attained.

A minimum principle given by Ponter and Chen (2001), capable to characterize the steady state response (or steady cycle) of an elastic perfectly plastic structure, has been here extended to cope with generalized standard materials so obtaining a variational tool to solve the above equation set.

A wider insight into this steady cycle minimum principle has been here achieved by a discussion of the related Euler–Lagrange equations using the classical Lagrange multiplier method; by this way, the effective capability of the minimum principle to solve the steady cycle problem has been assessed. The physical content of the minimum principle has been made clearer through the recognition that the quantity to minimize has the meaning of *total net resistant work* (equal to the difference between the total intrinsic dissipation energy and the work correspondingly done by the applied load), which is offered by the structure in an imposed CAS. The latter solution is constituted by kinematic and static parts known to play specific roles in the kinematic and static shakedown theorems, respectively; the minimum operation is to be achieved within the domain of all such solutions. Additionally, the minimum principle has been recognized to be a time/space integrated form of the so-called *minimum net resistant power theorem*—here established—which is dual of the maximum intrinsic dissipation theorem of plasticity theory and provides the plastic flow mechanism (if any) corresponding to an assigned stress/hardening state of the material.

The above minimum principle has been also discussed for the particular case of loads not exceeding the shakedown limit. With a procedure that seems more rigorous than that used by Ponter and Chen (2001), it has been shown that in the considered case the minimum principle decouples into two pieces, one of which is a static-type search problem conforming to the Melan theorem of shakedown theory, the other is a Markovian kinematic minimum principle for the incipient collapse mode, equivalent to the Koiter theorem of shakedown theory. This circumstance, together with the nature of the constituent kinematic and static parts of a generic CAS, made it possible to interpret the steady cycle minimum principle as a special combined form of either static and kinematic shakedown theorems, valid for loads in excess to the shakedown limit, but which for loads not exceeding this limit decouples into the two separate shakedown statements. This result widens that given by Ponter and Chen (2001), who interpreted the steady cycle minimum principle as just an extension of the upper bound shakedown theorem to loads in excess to the shakedown limit.

Another particular case to which the steady cycle minimum principle has been applied is that of a material element (or specimen) subjected to a given cyclic uniform stress (or strain). The simpler form taken on by this minimum principle, characterized by the disappearance of the space co-ordinates, has been found by means of a direct study of the problem in order to better clarify the physical and mathematical content of it.

Using the steady cycle minimum principle, criteria for the assessment of the ratchet limit loads have been formulated. These criteria are useful to discern loads below the ratchet limit (which thus produce either alternating plasticity, or shakedown) from loads exceeding this limit (hence producing ratchetting).

A technically relevant aspect of the entire theory is the possibility there offered to determine, for a given structure, cyclic loads at the ratchet limit and, in particular in the case of a two-parameter family of cyclic/steady loads, the border line separating, in the Bree-like diagram, the ratchetting zone from the no ratchetting one. The procedure to evaluate the ratchet limit loads was explained by Ponter and Chen (2001) and Chen and Ponter (2001) for perfect plasticity. Here, the same procedure has been reproposed for generalized standard materials, and additionally a deeper insight on the conditions upon which this procedure grounds has been provided.

Essentially, this procedure exploits the insensitivity features of the structure being in an alternating plasticity condition with respect to permanent (mechanical) loads applied upon the (nonempty) elastic region V_e, which in fact leave unaltered the existing steady cycle in all, except for a stress increment in V_e equilibrating the applied load (Polizzotto, 1993a, 1994a,b). Thus, the procedure needs some cyclic load not exceeding the ratchet limit to be used as a master load for generating the relevant alternating plasticity steady cycle, and determines the maximum permanent load amplitude for which these insensitivity features persist. Therefore, like in Ponter and Chen (2001), the procedure consists in two steps:

(i) Using the steady cycle minimum principle with a master load, determine the relevant alternating plasticity steady cycle. This master load can be chosen coincident with the given cyclic load if the latter is known (or recognized by the appropriate criteria) not to exceed the ratchet limit, but can be properly obtained (in ways that have been here suggested) from the given cyclic load if the latter exceeds the ratchet limit.
(ii) Determine the maximum amplitude of the permanent load that can be applied upon V_e in the body being in the same alternating plasticity state previously evaluated.

An aspect of this two-step procedure, here pointed out, is its unified character for cyclic loads of different amplitude, that is: below the alternating plasticity limit, in which case the master load taken alone is a shakedown load, whereas, taken in combination with the permanent load of maximum amplitude, forms a shakedown limit load; above the alternating plasticity limit, in which case the body $V_e \subset V$ considered isolated from the rest of the body finds itself in a condition of shakedown, Ponter and Karadenitz (1985), Polizzotto (1993a, 1994b).

The contribution given in this paper mainly consists in an extension and a generalization of previous results due to Gokhfeld and Cherniavsky (1980) and to Ponter and Chen (2001) in the domain of perfect plasticity. But it—the author believes—also provides a deeper insight into the inherent theory and contains original additions and nonsecondary improvements like, among other: a maximum principle here devised to characterize the optimal initial stress/hardening state in the steady cycle, a clearer relationship between the governing equation set and the steady cycle minimum principle, the very mechanical roots of the latter principle in the material behavior through the here envisioned minimum net resistant power theorem, the assessment of the mechanical implications of the steady cycle minimum principle through the study of the related Euler–Lagrange equations, a more rigorous passage to the limit for the steady cycle minimum principle in the presence of loads at the shakedown limit, the formulation of criteria for the ratchet limit loads, the assessment of ground motivations for the two-step procedure for the ratchet limit analysis.

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Appendix A. Notation

A compact notation is used, with bold-face letters for vectors and tensors. The *dot* and *colon* products between vectors and tensors denote the simple and double index contraction operations, respectively. For instance, considering the vectors $\boldsymbol{u} = \{u_i\}$, $\boldsymbol{v} = \{v_i\}$, $\boldsymbol{n} = \{n_i\}$ and the tensors $\boldsymbol{\sigma} = \{\sigma_{ij}\}$, $\boldsymbol{\varepsilon} = \{\varepsilon_{ij}\}$ and $\boldsymbol{D} = \{D_{ijhk}\}$, one can write: $\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i$, $\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ji}$, $\boldsymbol{n} \cdot \boldsymbol{\sigma} = \{n_j \sigma_{ji}\}$, $\boldsymbol{D} : \boldsymbol{\varepsilon} = \{D_{ijhk}\varepsilon_{kh}\}$, where the subscripts denote Cartesian components and the repeated index summation rule is applied. Cartesian orthogonal coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ are employed. The symbol ∇^s denotes the symmetric part of the gradient operator, i.e. $\nabla^s \boldsymbol{u} = [\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T]/2$. The symbol $:=$ means equality by definition. The right hand side of $\min F(\cdot) := [\ldots]$ specifies the objective function/functional $F(\cdot)$, the optimal value of which, F^*, is denoted a $F^* = \min F(\cdot)$, or $\min F(\cdot) \rightarrow F^*$. Other symbols are defined in the text at the place of their first appearance.

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